The volume inequality in Ehrhart's conjecture

Abstract

Let $K \subset \mathbb{R}^n$ be a convex body whose centroid is a lattice point and is the only lattice point in the interior of K. We prove that $\mathrm{vol}(K) \le (n+1)^n/n!$. The proof is complex-analytic: on the algebraic torus $T = (\mathbb{C}^*)^n$ the barycenter hypothesis is used once, through the existence of the toric Kähler–Einstein (Monge–Ampère) potential whose gradient image is K; the lattice hypothesis is used once, through the rank-one property of the associated weighted Bergman space. Rank-one-ness, via Berndtsson's subharmonicity theorem, makes a Ding-type functional convex along a ray of envelopes with prescribed Lelong number at a point; a volume estimate caps the asymptotic slope of that functional, while a comparison-principle estimate for the Monge–Ampère masses of the envelopes bounds the contact measure from below. Together these close the chain $\frac{n}{n+1}\tau \le n$, where $\tau = (n!\mathrm{vol}K)^{1/n}$.

1 Introduction

Theorem 1.1 (Ehrhart's volume inequality)**.** *Let $K \subset \mathbb{R}^n$ be a convex body whose centroid $g(K)$ lies in $\mathbb{Z}^n$ and is the only point of $\mathbb{Z}^n$ in the interior of K. Then*

$$\mathrm{vol}(K) \;\le\; \frac{(n+1)^n}{n!}.$$

Translating by $-g(K) \in \mathbb{Z}^n$ we may and do normalize $g(K) = 0$; the hypotheses become

$$\text{(i)}\quad b(K) := \frac{1}{\mathrm{vol}K}\int_K y\,dy = 0, \qquad\qquad \text{(ii)}\quad \mathrm{int}K \cap \mathbb{Z}^n = \{0\}.$$

Thus Theorem 1.1 is equivalent to the following statement, which is what we prove.

Theorem 1.2. *Let $K \subset \mathbb{R}^n$ be a convex body with $b(K) = 0$ and $\mathrm{int}K \cap \mathbb{Z}^n = \{0\}$. Then*

$$n!\,\mathrm{vol}(K) \;\le\; (n+1)^n.$$

Outline. The whole argument lives on the algebraic torus $T = (\mathbb{C}^*)^n$, with logarithm map $L(z) = (\log|z_1|^2, \dots, \log|z_n|^2)$.

- Hypothesis (i) is consumed exactly once (§3): it guarantees the existence of a smooth strictly convex ϕ on $\mathbb{R}^n$ solving the toric Kähler–Einstein equation $\det D^2\phi = e^{-\phi}$ with $\overline{\nabla\phi(\mathbb{R}^n)} = K$. Setting $\Phi_0 := \phi \circ L$ this becomes the identity $(i\partial\bar\partial\Phi_0)^n = n!\,e^{-\Phi_0}\mu_\Omega$ on T, where μ_Ω is the invariant measure.

- Hypothesis (ii) is consumed exactly once (§4): it forces the weighted Bergman space $\mathcal{H}_{\Phi_0} = \{f \in \mathcal{O}(T) : \int |f|^2 e^{-\Phi_0}\mu_\Omega < \infty\}$ to be $\mathbb{C}\cdot 1$.

- For $\lambda \geq 0$ let P_λ be the largest plurisubharmonic function $\leq \Phi_0$ with Lelong number $\geq \lambda$ at $p = (1, \ldots, 1)$, let $C_\lambda = \{P_\lambda = \Phi_0\}$ be its contact set, and let $\lambda^*(z) = \sup\{\lambda < \tau : z \in C_\lambda\}$, where $\tau := (n!\mathrm{vol}K)^{1/n}$ (§5).

- Rank-one-ness plus Berndtsson's subharmonicity theorem make the Ding-type functional $g(s) = -\log \int_T e^{-\Psi_s} \mu_\Omega$ convex along the ray Ψ_s of Legendre-type envelopes built from the P_λ (§6); a volume estimate inside a shrinking ball at p gives $g(s) \leq -(\tau - n)s + C$. Combining, $\int_T \lambda^* \, d\beta \leq n$, where β is the normalized measure $e^{-\Phi_0}\mu_\Omega / \int e^{-\Phi_0}\mu_\Omega$.

- A comparison-principle argument (§7) shows that the non-contact set is small: $\mathfrak{m}(\{P_\lambda < \Phi_0\}) \leq \lambda^n/n!$, where $\mathfrak{m} := \frac{1}{(2\pi)^n n!}(i\partial\bar\partial\Phi_0)^n$. By the layer-cake formula this gives $\int_T \lambda^* \, d\mathfrak{m} \geq \frac{n}{n+1}\tau\mathrm{vol}(K)$.

- Since $\mathfrak{m} = \mathrm{vol}(K)\beta$, the two bounds give $\frac{n}{n+1}\tau \leq n$, i.e. $\tau \leq n+1$ (§8).

Classical results used. We use as cited black boxes: the existence theorem for moment measures of Cordero-Erausquin–Klartag [8], building on Berman–Berndtsson [3] and Wang–Zhu [13] (Theorem 3.1); Caffarelli's strict convexity and interior regularity theory for the Monge–Ampère equation [5, 6, 7] (Proposition 3.2); Berndtsson's theorem on plurisubharmonicity of weighted Bergman kernels [4] (Theorem 6.3); the Bedford–Taylor calculus (comparison principle, decreasing convergence, plurifine locality, Perron–Bremermann envelopes on balls) [1, 2, 9, 10, 12]; and the Green function of a plane annulus [11].

2 Notation and pluripotential preliminaries

Throughout, $T := (\mathbb{C}^*)^n$, $p := (1, \ldots, 1) \in T$, and

$$L : T \to \mathbb{R}^n, \qquad L(z) := (\log|z_1|^2, \ldots, \log|z_n|^2), \qquad x := L(z).$$

With $\Omega := \bigwedge_{j=1}^n \frac{dz_j}{z_j}$ we set $\mu_\Omega := i^{n^2}\Omega \wedge \overline{\Omega}$. In polar coordinates $z_j = e^{x_j/2 + i\theta_j}$,

$$\mu_\Omega = 2^n \prod_j \frac{dV_j}{|z_j|^2} = dx \otimes d\theta, \qquad \theta \in [0, 2\pi)^n, \tag{1}$$

where dV denotes Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. (For $n = 1$: $i\,\frac{dz \wedge d\bar z}{|z|^2} = \frac{2\,dV}{|z|^2} = 2\,d(\log r)\,d\theta = dx\,d\theta$.)

For $u \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}$ we write $(i\partial\bar\partial u)^n$ for the Bedford–Taylor Monge–Ampère measure and set

$$\mathfrak{m}_u := \frac{1}{(2\pi)^n\, n!}\,(i\partial\bar\partial u)^n. \tag{2}$$

Recall that for a C^2 function u one has $(i\partial\bar\partial u)^n = n!\,2^n \det\left(u_{j\bar k}\right) dV$.

Lemma 2.1 (Invariant dictionary). *If $\psi : \mathbb{R}^n \to \mathbb{R}$ is convex and smooth and $\Psi = \psi \circ L$, then*

$$(i\partial\bar\partial\Psi)^n = n!\,\det D^2\psi(L(z))\,dx\,d\theta,$$

hence $L_*\mathfrak{m}_\Psi = \det D^2\psi\,dx$, the real Monge–Ampère measure of ψ.

Proof. Locally $L_j = \log z_j + \log \bar z_j$, so $\Psi_{j\bar k} = \psi_{jk}(L)/(z_j \bar z_k)$ and therefore $(i\partial\bar\partial\Psi)^n = n!\,2^n \det(\psi_{jk}(L)) \prod_j |z_j|^{-2}\,dV$. Now apply (1). Integrating out θ contributes the factor $(2\pi)^n$, which cancels the normalization in (2). $\square$

Lemma 2.2 (Stokes mass equality). *Let $B \subset \mathbb{C}^n$ be a ball and let $u, v \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}$ on a neighbourhood of $\overline{B}$ with $u = v$ on a neighbourhood N of ∂B. Then*

$$\int_{\overline{B}} (i\partial\bar\partial u)^n = \int_{\overline{B}} (i\partial\bar\partial v)^n.$$

Proof. Choose a slightly larger concentric ball $B' \supset \overline{B}$ with $\overline{B'} \setminus B \subset N$, and $\chi \in C^\infty_c(B')$ with $\chi \equiv 1$ on $\overline{B}$; then $\operatorname{supp} d\chi \subset B' \setminus \overline{B} \subset N$. By locality of the Monge–Ampère operator, $(i\partial\bar\partial u)^n = (i\partial\bar\partial v)^n$ on N, so

$$\int \chi\big[(i\partial\bar\partial u)^n - (i\partial\bar\partial v)^n\big] = \int_{\overline{B}} \big[(i\partial\bar\partial u)^n - (i\partial\bar\partial v)^n\big].$$

On the other hand, writing $(i\partial\bar\partial u)^n - (i\partial\bar\partial v)^n = i\partial\bar\partial(u - v) \wedge \mathcal{T}$ with $\mathcal{T} := \sum_{k=0}^{n-1} (i\partial\bar\partial u)^k \wedge (i\partial\bar\partial v)^{n-1-k}$ a closed positive current, two integrations by parts (legitimate for locally bounded plurisubharmonic functions, [1]) give

$$\int \chi\, i\partial\bar\partial(u - v) \wedge \mathcal{T} = \int (u - v)\, i\partial\bar\partial\chi \wedge \mathcal{T} = 0,$$

because $u - v \equiv 0$ on $\operatorname{supp} d\chi$. $\square$

Lemma 2.3 (Model masses). *(a) $\big(i\partial\bar\partial \log(|z|^2 + \varepsilon^2)\big)^n = n!\,2^n \dfrac{\varepsilon^2}{(|z|^2 + \varepsilon^2)^{n+1}}\,dV$, a measure of total mass $(2\pi)^n$ on $\mathbb{C}^n$, whose mass concentrates at the origin as $\varepsilon \downarrow 0$.*

(b) For $s > 0$ let $v_s := \max(\log |z|^2, -s)$. Then $(i\partial\bar\partial v_s)^n$ is supported on the sphere $\{|z| = e^{-s/2}\}$ and has total mass $(2\pi)^n$.

(c) Consequently, for $\lambda > 0$, $a \in \mathbb{C}^n$ and $s > 0$, the truncated model $\lambda \max(\log |z - a|^2, -s)$ has Monge–Ampère mass $\lambda^n (2\pi)^n$, carried by the sphere $\{|z - a| = e^{-s/2}\}$; in the normalization (2) its $\mathfrak{m}$-mass is $\lambda^n/n!$.

Proof. (a) With $u = \log(|z|^2 + \varepsilon^2)$ one has $u_{j\bar k} = \frac{\delta_{jk}}{|z|^2+\varepsilon^2} - \frac{\bar z_j z_k}{(|z|^2+\varepsilon^2)^2}$, whose determinant equals $\varepsilon^2(|z|^2 + \varepsilon^2)^{-(n+1)}$; this gives the stated density. Substituting $z = \varepsilon w$,

$$\int_{\mathbb{C}^n} \frac{n!\,2^n \varepsilon^2\, dV(z)}{(|z|^2 + \varepsilon^2)^{n+1}} = n!\,2^n \int_{\mathbb{C}^n} \frac{dV(w)}{(1 + |w|^2)^{n+1}} = n!\,2^n \cdot \frac{2\pi^n}{(n-1)!} \cdot \frac{1}{2n} = (2\pi)^n,$$

using $\mathrm{Vol}(S^{2n-1}) = 2\pi^n/(n-1)!$ and $\int_0^\infty r^{2n-1}(1 + r^2)^{-n-1} dr = \frac{1}{2n}$. The same substitution shows that for every $\rho > 0$ the mass outside $B(0, \rho)$ tends to 0 as $\varepsilon \downarrow 0$.

(b) v_s is plurisubharmonic and locally bounded; on $\{|z| < e^{-s/2}\}$ it is constant, and on $\{|z| > e^{-s/2}\}$ it equals $\log |z|^2$, whose complex Hessian is degenerate, so $(i\partial\bar\partial \log |z|^2)^n = 0$ there. By locality $(i\partial\bar\partial v_s)^n$ is carried by the sphere $\{|z| = e^{-s/2}\}$.

For the mass, put $v_s^\varepsilon := \max(\log(|z|^2 + \varepsilon^2), -s)$. For ε small, $v_s^\varepsilon = \log(|z|^2 + \varepsilon^2)$ near $\partial B(0, 1)$, so Lemma 2.2 and (a) give $\int_{\overline{B}(0,1)} (i\partial\bar\partial v_s^\varepsilon)^n = \int_{\overline{B}(0,1)} (i\partial\bar\partial \log(|z|^2 + \varepsilon^2))^n \to (2\pi)^n$ as $\varepsilon \downarrow 0$.

Moreover, by locality $(i\partial\bar\partial v_s^\varepsilon)^n$ equals the measure in (a) on $\{\log(|z|^2+\varepsilon^2) > -s\} \supset \{|z| > e^{-s/2}\}$, and by (a) the mass of that part tends to 0. Fix $\chi \in C_c(B(0,1))$ with $0 \le \chi \le 1$ and $\chi \equiv 1$ on a neighbourhood of $\{|z| = e^{-s/2}\}$; then $\int \chi(i\partial\bar\partial v_s^\varepsilon)^n \to (2\pi)^n$. Since $v_s^\varepsilon \downarrow v_s$ with all functions locally bounded, Bedford–Taylor decreasing convergence [1] yields $\int \chi\,(i\partial\bar\partial v_s)^n = (2\pi)^n$, and since $(i\partial\bar\partial v_s)^n$ is carried by $\{|z| = e^{-s/2}\}$, this is its total mass.

(c) Translate, and use $(i\partial\bar\partial(\lambda u))^n = \lambda^n(i\partial\bar\partial u)^n$. □

Lelong numbers. For u plurisubharmonic near p set $M_u(r) := \sup_{|z-p|=r} u$. As is classical, $t \mapsto M_u(e^{t/2})$ is convex and nondecreasing. Define

$$\nu_p(u) := \sup\{c \ge 0 :\ u \le c\log|z-p|^2 + O(1) \text{ near } p\}.$$

Lemma 2.4 (Slope lemma)**.** *Let u be plurisubharmonic on a neighbourhood of $\overline{B}(p, r_0)$ with $u \le M_0$ there. If $\nu_p(u) \ge \lambda$, then*

$$u(z) \ \le\ \lambda\log\frac{|z-p|^2}{r_0^2} + M_0 \qquad (|z-p| \le r_0). \tag{3}$$

Conversely, (3) *implies $\nu_p(u) \ge \lambda$. Moreover $\nu_p(\lambda\log|z-p|^2) = \lambda$ and $\nu_p\big((1-t)u_0 + tu_1\big) \ge (1-t)\nu_p(u_0) + t\nu_p(u_1)$ for $t \in [0,1]$.*

Proof. Write $m(t) := M_u(e^{t/2})$, a convex nondecreasing function of $t = \log r^2$, and $t_0 := \log r_0^2$. Let $c < \lambda$. By definition of ν_p there is C with $m(t) \le ct + C$ for all t small enough. If some slope of m were $< c$, convexity would force $m(t) \ge m(t_1) + \alpha(t-t_1)$ with $\alpha < c$ for all $t \le t_1$, contradicting $m(t) \le ct + C$ as $t \to -\infty$. Hence all slopes of m are $\ge c$, and so $m(t) \le m(t_0) + c(t-t_0) \le M_0 + c(t-t_0)$ for $t \le t_0$. Since $t - t_0 \le 0$, letting $c \uparrow \lambda$ gives (3). The converse and the last two statements are immediate from the definition (for the convex combination, add the two $O(1)$ bounds). □

Finally, we record the standard facts that will be used without further comment: the regularized supremum $(\sup_\alpha u_\alpha)^*$ of a family of plurisubharmonic functions locally uniformly bounded above is plurisubharmonic (or $\equiv -\infty$), and it coincides with $\sup_\alpha u_\alpha$ outside a pluripolar (in particular Lebesgue-null) set; a decreasing limit of plurisubharmonic functions on a connected open set is plurisubharmonic or $\equiv -\infty$; and if u, v are plurisubharmonic and $u \ge v$ almost everywhere, then $u \ge v$ everywhere.

3 The Monge–Ampère potential

Theorem 3.1 (Wang–Zhu; Berman–Berndtsson; Cordero-Erausquin–Klartag)**.** *Let $K \subset \mathbb{R}^n$ be a convex body with $b(K) = 0$ (hence $0 \in \mathrm{int}K$). There exists a convex function $\phi : \mathbb{R}^n \to \mathbb{R}$, an Alexandrov solution of*

$$\det D^2\phi = e^{-\phi} \ \text{ on } \mathbb{R}^n, \qquad \overline{\nabla\phi(\mathbb{R}^n)} = K,$$

meaning that the Monge–Ampère measure $\mathrm{MA}(\phi)(E) := |\partial\phi(E)|$ equals $e^{-\phi}dx$ and $(\nabla\phi)_*\big(e^{-\phi}dx\big) = \mathbf{1}_K\,dy$. It is unique up to translations of the variable. Conversely, solvability forces $b(K) = 0$.

References. This is the moment measure theorem of Cordero-Erausquin–Klartag [8] applied to $\mu = \mathbf{1}_K\, dy$ (a finite measure with barycenter 0, not supported in a hyperplane); the method is the variational one of Berman–Berndtsson [3], descending from Wang–Zhu [13]. Variationally, one minimizes $J(\phi) = \frac{1}{\mathrm{vol} K}\int_K \phi^*\, dy + \log \int_{\mathbb{R}^n} e^{-\phi} dx$ over convex ϕ with ϕ^* finite on K; J is invariant under translations precisely because $b(K) = 0$, is convex by Prékopa's theorem, and is coercive after a John normalization; its Euler–Lagrange equation is the one stated. For the necessity of $b(K) = 0$, integrate $\nabla(e^{-\phi})$ over $\mathbb{R}^n$: $0 = \int \nabla\phi\, e^{-\phi} dx = \int_K y\, dy = \mathrm{vol}(K)\, b(K)$. $\square$

Proposition 3.2 (Regularity)**.** *The function ϕ of Theorem 3.1 is smooth and strictly convex, with $D^2\phi > 0$.*

Proof. ϕ is finite and convex on $\mathbb{R}^n$, hence locally Lipschitz, so the right-hand side $f := e^{-\phi}$ is locally Lipschitz and locally pinched between positive constants. Suppose ϕ were not strictly convex: then ϕ coincides with some supporting affine function ℓ on a closed convex contact set W containing more than one point. By Caffarelli's localization theorem [5], W has no extreme point in the interior of the domain, which here is all of $\mathbb{R}^n$; a closed convex set with more than one point and no extreme point contains a full line $x_0 + \mathbb{R}v$. Then for any $x \in \mathbb{R}^n$ and $y \in \partial\phi(x)$,

$$\ell(x_0 + tv) = \phi(x_0 + tv) \;\geq\; \phi(x) + \langle y, x_0 + tv - x\rangle \qquad (t \in \mathbb{R}),$$

with both sides affine in t; comparing slopes forces $\langle y, v\rangle = \langle \nabla\ell, v\rangle$ for every subgradient y at every point. Thus $\nabla\phi(\mathbb{R}^n)$ lies in a hyperplane, contradicting $\overline{\nabla\phi(\mathbb{R}^n)} = K$, which is full-dimensional. Hence ϕ is strictly convex, and Caffarelli's interior regularity theory [6, 7] gives $\phi \in C^{2,\alpha}_{\mathrm{loc}}$; bootstrapping the now uniformly elliptic equation gives $\phi \in C^\infty$. Finally $\det D^2\phi = e^{-\phi} > 0$ together with $D^2\phi \geq 0$ gives $D^2\phi > 0$. $\square$

From now on K is as in Theorem 1.2, ϕ is the potential of Theorem 3.1,

$$V := \mathrm{vol}(K), \qquad \tau := (n!\, V)^{1/n}, \qquad \Phi_0 := \phi \circ L.$$

The goal of the paper is the inequality $\tau \leq n + 1$.

Lemma 3.3. *(a)* $\int_{\mathbb{R}^n} e^{-\phi}\, dx = V$.

(b) $\phi(x) \leq \phi(0) + h_K(x)$ *for all* x*, where* h_K *is the support function of* K*.*

(c) $\Phi_0 \in C^\infty(T)$ *is strictly plurisubharmonic, and with* $\mathfrak{m} := \mathfrak{m}_{\Phi_0}$ *as in* (2)*,*

$$\mathfrak{m} = \frac{1}{(2\pi)^n} e^{-\Phi_0}\mu_\Omega, \qquad \mathfrak{m}(T) = V, \qquad \int_T e^{-\Phi_0}\mu_\Omega = (2\pi)^n V. \tag{4}$$

Proof. (a) Compare total masses in $(\nabla\phi)_*(e^{-\phi}dx) = \mathbf{1}_K dy$. (b) $\phi(x) - \phi(0) = \int_0^1 \langle\nabla\phi(tx), x\rangle\, dt \leq h_K(x)$, because $\nabla\phi \in K$. (c) By Lemma 2.1 and the equation, $(i\partial\bar\partial\Phi_0)^n = n!\, e^{-\phi(L)} dx\, d\theta = n!\, e^{-\Phi_0}\mu_\Omega$; strict plurisubharmonicity follows from $D^2\phi > 0$. The total mass is $\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{[0,2\pi)^n} e^{-\phi(x)} dx\, d\theta = V$ by (a). $\square$

We write $\beta := \mathfrak{m}/V$, a probability measure on T; explicitly $d\beta = e^{-\Phi_0}\mu_\Omega / \big((2\pi)^n V\big)$.

4 The rank-one property

For ψ plurisubharmonic on T put

$$\mathcal{H}_\psi := \Big\{ f \in \mathcal{O}(T) : \ \|f\|_\psi^2 := \int_T |f|^2 e^{-\psi} \mu_\Omega < \infty \Big\},$$

and for $m \in \mathbb{Z}^n$ set $J(m) := \int_{\mathbb{R}^n} e^{\langle m,x\rangle - \phi(x)} dx$.

Lemma 4.1. *$J(0) = V < \infty$, and $J(m) = \infty$ for every $m \in \mathbb{Z}^n \setminus \mathrm{int} K$.*

Proof. The first claim is Lemma 3.3(a). For the second, by Lemma 3.3(b) and $h_{K-m}(x) = h_K(x) - \langle m, x\rangle$,

$$e^{\langle m,x\rangle - \phi(x)} \ \geq \ e^{-\phi(0)} e^{-h_{K'}(x)}, \qquad K' := K - m.$$

Since $m \notin \mathrm{int}K$ we have $0 \notin \mathrm{int}K'$, so there is a unit vector u with $\langle u, y\rangle \leq 0$ for all $y \in K'$, i.e. $h_{K'}(u) \leq 0$. On the set $S := \{tu + w : \ t \geq 0, \ |w| \leq 1\}$, subadditivity and positive homogeneity of $h_{K'}$ give

$$h_{K'}(tu + w) \ \leq \ t\, h_{K'}(u) + h_{K'}(w) \ \leq \ \max_{|w|\leq 1} h_{K'}(w) =: c < \infty.$$

Thus the integrand is bounded below by $e^{-\phi(0)-c} > 0$ on S, a set of infinite Lebesgue measure; hence $J(m) = \infty$. □

Lemma 4.2 (Rank one)**.** *If $\mathrm{int}K \cap \mathbb{Z}^n = \{0\}$, then $\mathcal{H}_{\Phi_0} = \mathbb{C} \cdot 1$.*

Proof. Every $f \in \mathcal{O}(T)$ has a Laurent expansion $f = \sum_{m\in\mathbb{Z}^n} c_m z^m$, converging locally uniformly. Let $E \subset \mathbb{R}^n$ be compact. Since the weight $e^{-\Phi_0}$ is independent of θ and $|z^m|^2 = e^{\langle m,x\rangle}$, Parseval's identity in θ and (1) give

$$\int_{L^{-1}(E)} |f|^2 e^{-\Phi_0} \mu_\Omega = (2\pi)^n \sum_{m\in\mathbb{Z}^n} |c_m|^2 \int_E e^{\langle m,x\rangle - \phi(x)} dx.$$

Letting $E \uparrow \mathbb{R}^n$ and using monotone convergence, $\|f\|_{\Phi_0}^2 = (2\pi)^n \sum_m |c_m|^2 J(m)$. If this is finite then $c_m = 0$ whenever $J(m) = \infty$, i.e. (by Lemma 4.1 and the hypothesis) for all $m \neq 0$. Conversely $\|1\|_{\Phi_0}^2 = (2\pi)^n V < \infty$. □

5 Envelopes with prescribed Lelong number

For an open set $\Omega \subseteq T$ with $p \in \Omega$ put

$$\mathcal{F}_\lambda(\Omega) := \{\Gamma \in \mathrm{PSH}(\Omega) : \ \Gamma \leq \Phi_0, \ \nu_p(\Gamma) \geq \lambda\}, \qquad P_\lambda^\Omega := \Big(\sup_{\Gamma \in \mathcal{F}_\lambda(\Omega)} \Gamma \Big)^*,$$

with the convention $\sup \emptyset := -\infty$, and write $P_\lambda := P_\lambda^T$. Fix once and for all

$$r_0 \in (0, \tfrac{1}{2}] \ \text{ with } \ \overline{B}(p, r_0) \subset T, \qquad M_0 := \sup_{B(p,r_0)} \Phi_0 < \infty.$$

Lemma 5.1. *(a) $P_0 = \Phi_0$; $P_\lambda^\Omega \leq \Phi_0$, and $\lambda \mapsto P_\lambda^\Omega$ is nonincreasing.*

(b) *(Uniform singularity bound) Every* $\Gamma \in \mathcal{F}_\lambda(\Omega)$*, and hence* P_λ^Ω *itself, satisfies*

$$\Gamma(z) \ \leq \ \lambda \log \frac{|z-p|^2}{r_0^2} + M_0 \qquad \text{on } B(p, r_0) \cap \Omega.$$

In particular $\nu_p(P_\lambda^\Omega) \geq \lambda$*, and* $P_\lambda^\Omega \in \mathcal{F}_\lambda(\Omega)$ *whenever* $P_\lambda^\Omega \not\equiv -\infty$*.*

(c) *(Concavity) For fixed* z*,* $\lambda \mapsto P_\lambda^\Omega(z)$ *is concave on* $[0, \infty)$ *(values in* $[-\infty, \infty)$*).*

(d) *(Interval property) If* $P_{\lambda_1}(z) = \Phi_0(z)$ *and* $0 \leq \lambda \leq \lambda_1$*, then* $P_\lambda(z) = \Phi_0(z)$*.*

Proof. (a) $\Phi_0 \in \mathcal{F}_0(T)$; the rest is clear, the bound $P_\lambda^\Omega \leq \Phi_0$ surviving the regularization because Φ_0 is continuous.

(b) Each Γ satisfies $\Gamma \leq \Phi_0 \leq M_0$ on $B(p, r_0) \cap \Omega$, so Lemma 2.4 applies (after shrinking the ball if $\Omega \neq T$, which only improves the constant; for $\Omega \supseteq B(p, r_0)$ it applies directly). The majorant is continuous, hence the bound passes to the supremum and to its regularization. The converse part of Lemma 2.4 gives $\nu_p(P_\lambda^\Omega) \geq \lambda$.

(c) Let $S_\lambda := \sup_{\Gamma \in \mathcal{F}_\lambda(\Omega)} \Gamma$, so $P_\lambda^\Omega = S_\lambda^*$. If $\Gamma_i \in \mathcal{F}_{\lambda_i}(\Omega)$ and $t \in [0,1]$, then $(1-t)\Gamma_0 + t\Gamma_1 \in \mathcal{F}_{(1-t)\lambda_0 + t\lambda_1}(\Omega)$ by Lemma 2.4; taking suprema, $S_{(1-t)\lambda_0 + t\lambda_1} \geq (1-t)S_{\lambda_0} + tS_{\lambda_1}$ pointwise. Outside a Lebesgue-null set $S_{\lambda_i} = P_{\lambda_i}^\Omega$ $(i = 0, 1)$, so $P_{(1-t)\lambda_0 + t\lambda_1}^\Omega \geq (1-t)P_{\lambda_0}^\Omega + tP_{\lambda_1}^\Omega$ almost everywhere; both sides being plurisubharmonic (or $\equiv -\infty$), the inequality holds everywhere.

(d) Apply (c) with $\lambda = (1 - \frac{\lambda}{\lambda_1}) \cdot 0 + \frac{\lambda}{\lambda_1}\lambda_1$: $P_\lambda(z) \geq (1 - \frac{\lambda}{\lambda_1})\Phi_0(z) + \frac{\lambda}{\lambda_1}P_{\lambda_1}(z) = \Phi_0(z)$, and $P_\lambda \leq \Phi_0$. □

Definition 5.2. The *contact set* is $C_\lambda := \{z \in T : \ P_\lambda(z) = \Phi_0(z)\}$; it is closed, since $P_\lambda - \Phi_0$ is upper semicontinuous and $C_\lambda = \{P_\lambda - \Phi_0 \geq 0\}$. The *contact threshold* is

$$\lambda^*(z) := \sup\{\lambda \in [0, \tau) : \ z \in C_\lambda\} \in [0, \tau] \qquad (\sup \emptyset := 0).$$

Note $C_0 = T$, so the supremum is over a nonempty set and $\lambda^* \geq 0$.

Lemma 5.3. λ^* *is Borel measurable, and* $\{\lambda^* > \lambda\} \subseteq C_\lambda \subseteq \{\lambda^* \geq \lambda\}$ *for* $\lambda \in (0, \tau)$*. Consequently, for every finite Borel measure* μ *on* T*,*

$$\int_T \lambda^* \, d\mu = \int_0^\tau \mu(C_\lambda) \, d\lambda.$$

Proof. By Lemma 5.1(d), $\lambda^*(z) = \sup\{\lambda \in \mathbb{Q} \cap [0, \tau) : z \in C_\lambda\}$, so $\{\lambda^* > c\} = \bigcup_{\lambda \in \mathbb{Q} \cap (c, \tau)} C_\lambda$ is Borel. The stated inclusions are again Lemma 5.1(d) and the definition of λ^*. Since $0 \leq \lambda^* \leq \tau$, the layer-cake formula gives

$$\int_T \lambda^* d\mu = \int_0^\tau \mu(\{\lambda^* > \lambda\}) \, d\lambda = \int_0^\tau \mu(\{\lambda^* \geq \lambda\}) \, d\lambda,$$

and the two ends sandwich $\int_0^\tau \mu(C_\lambda) d\lambda$. □

6 The ray and the convexity engine

Construction 6.1. On $S := \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > 0\} \times T$ define

$$U(\zeta, z) := \sup_{\lambda \in [0,\tau]} \Big(P_\lambda(z) + \operatorname{Re}\zeta \cdot (\lambda - \tau) \Big), \qquad \Psi := U^*,$$

the joint upper semicontinuous regularization. Since U depends only on $s := \operatorname{Re}\zeta$, so does Ψ; we write $\Psi_s(z) := \Psi(\zeta, z)$ for any ζ with $\operatorname{Re}\zeta = s$.

Lemma 6.2. *(a) Ψ is plurisubharmonic on S, and each Ψ_s is plurisubharmonic on T.*

(b) $\Phi_0 - s\tau \le \Psi_s \le \Phi_0$; $s \mapsto \Psi_s(z)$ is convex and nonincreasing on $(0, \infty)$; and

$$\Psi_s(z) \;\ge\; \Phi_0(z) + s\big(\lambda^*(z) - \tau\big) \qquad (z \in T,\ s > 0). \tag{5}$$

(c) (Well bound) For every $s > 0$, $\Psi_s \le M_0 - s\tau$ on $B_s := \{|z - p| < r_0 e^{-s/2}\}$.

Proof. (a) Each function $(\zeta, z) \mapsto P_\lambda(z) + \operatorname{Re}\zeta\,(\lambda - \tau)$ is plurisubharmonic on S (a plurisubharmonic function of z plus a pluriharmonic function of ζ), and the family is locally uniformly bounded above by $\Phi_0(z)$. Hence $\Psi = U^*$ is plurisubharmonic or $\equiv -\infty$; the lower bound $U \ge \Phi_0 - s\tau$ (take $\lambda = 0$) excludes the latter and also shows that no slice Ψ_s is $\equiv -\infty$.

(b) The bounds $\Phi_0 - s\tau \le U \le \Phi_0$ hold pointwise; Φ_0 being continuous, $\Psi = U^* \le \Phi_0$, and $\Psi \ge U$ gives the lower bound. For convexity, fix $s_1, s_2 > 0$, $t \in (0, 1)$, $s_0 := (1 - t)s_1 + ts_2$ and $z_0 \in T$. For (s, z) near (s_0, z_0) put $s_i' := s_i + (s - s_0) > 0$; then $(1 - t)s_1' + ts_2' = s$, and since for fixed z the function $s \mapsto U(s, z)$ is a supremum of affine functions of s (with slopes $\lambda - \tau \le 0$), it is convex and nonincreasing, so

$$U(s, z) \le (1 - t)U(s_1', z) + t\,U(s_2', z).$$

Taking $\limsup$ as $(s, z) \to (s_0, z_0)$ (so that $(s_i', z) \to (s_i, z_0)$) yields $\Psi_{s_0}(z_0) \le (1 - t)\Psi_{s_1}(z_0) + t\Psi_{s_2}(z_0)$. Monotonicity is obtained in the same way from $U(s, z) \le U(s - (s_2 - s_1), z)$ for $s_1 < s_2$.

For (5): if $\lambda < \lambda^*(z)$ then $z \in C_\lambda$ by Lemma 5.1(d), i.e. $P_\lambda(z) = \Phi_0(z)$, whence $\Psi_s(z) \ge U(s, z) \ge \Phi_0(z) + s(\lambda - \tau)$. Let $\lambda \uparrow \lambda^*(z)$.

(c) Let $z \in B_s$. For every $\lambda \in [0, \tau]$, Lemma 5.1(b) gives

$$P_\lambda(z) + s(\lambda - \tau) \;\le\; \lambda\Big(\log\frac{|z - p|^2}{r_0^2} + s\Big) + M_0 - s\tau \;\le\; M_0 - s\tau,$$

because the bracket is ≤ 0 on B_s and $\lambda \ge 0$. Hence $U \le M_0 - s\tau$ on $\{(s, z) : z \in B_s\}$, a set which is open in $(0, \infty) \times T$; therefore the bound persists under the joint regularization. $\square$

Define

$$g(s) := -\log \int_T e^{-\Psi_s}\mu_\Omega \quad (s > 0), \qquad g(0) := -\log\big((2\pi)^n V\big).$$

By Lemma 6.2(b) and (4), $e^{-\Phi_0} \le e^{-\Psi_s} \le e^{s\tau}e^{-\Phi_0}$, so g is finite with

$$|g(s) - g(0)| \le s\tau; \tag{6}$$

in particular g is continuous at 0.

We shall use the following theorem of Berndtsson.

Theorem 6.3 (Berndtsson [4]). *Let $D \subseteq \mathbb{C}^m_\zeta \times \mathbb{C}^n_z$ be a pseudoconvex domain and let φ be plurisubharmonic on D. For ζ in the base let $K_\zeta(z,w)$ denote the Bergman kernel of the space of holomorphic functions on the fibre D_ζ that are square integrable with respect to $e^{-\varphi(\zeta,\cdot)}dV$. Then $\log K_\zeta(z,z)$ is plurisubharmonic (or $\equiv -\infty$) on D.*

Theorem 6.4 (Rank-one convexity). *Assume $\mathcal{H}_{\Phi_0} = \mathbb{C}\cdot 1$. Then g is convex on $[0,\infty)$.*

Proof. Step 1 (rank one along the ray). Fix $s > 0$. If $\|f\|^2_{\Psi_s} < \infty$ then, since $\Psi_s \le \Phi_0$, also $\|f\|_{\Phi_0} \le \|f\|_{\Psi_s} < \infty$, so $f \in \mathcal{H}_{\Phi_0} = \mathbb{C}\cdot 1$. Conversely $1 \in \mathcal{H}_{\Psi_s}$ because $\int e^{-\Psi_s}\mu_\Omega \le e^{s\tau}(2\pi)^n V < \infty$. Hence

$$\mathcal{H}_{\Psi_s} = \mathbb{C}\cdot 1 \qquad (s>0). \tag{7}$$

Step 2 (Berndtsson on an exhaustion). For $R > 0$ let

$$A_R := \{t \in \mathbb{C} :\ e^{-R} < |t| < e^R\}, \qquad D_R := A_R^n \Subset T,$$

a pseudoconvex domain. On the pseudoconvex product $D^{(R)} := \{\operatorname{Re}\zeta > 0\} \times D_R$ consider the plurisubharmonic weight

$$\varphi(\zeta, z) := \Psi(\zeta, z) + \sum_{i=1}^n \log|z_i|^2.$$

By (1), $e^{-\Psi_s}\mu_\Omega = 2^n e^{-\varphi}dV$, so for $s = \operatorname{Re}\zeta$,

$$\int_{D_R} |f|^2 e^{-\varphi(\zeta,\cdot)}dV = 2^{-n}\int_{D_R} |f|^2 e^{-\Psi_s}\mu_\Omega.$$

By Theorem 6.3, $\log K^{(R)}_\zeta(z,z)$ is plurisubharmonic on $D^{(R)}$, where $K^{(R)}_\zeta$ is the Bergman kernel of $A^2(D_R, e^{-\varphi(\zeta,\cdot)}dV)$; it depends only on $\operatorname{Re}\zeta$, since the weight and the domain do.

Step 3 (passing to the limit). Write $\|f\|^2_{D_R,\zeta} := \int_{D_R}|f|^2 e^{-\varphi(\zeta,\cdot)}dV$ and $\|f\|^2_{T,\zeta} := \int_T |f|^2 e^{-\varphi(\zeta,\cdot)}dV = 2^{-n}\|f\|^2_{\Psi_s}$. Since $K_\zeta(z_0,z_0) = \sup\{|f(z_0)|^2 :\ \|f\| \le 1\}$ and enlarging the domain only increases norms, we get for $R < R'$

$$K^{(R)}_\zeta(z_0,z_0) \ \ge\ K^{(R')}_\zeta(z_0,z_0) \ \ge\ K^T_\zeta(z_0,z_0),$$

where K^T is the Bergman kernel of $\mathcal{H}_{\Psi_s}$ with respect to $e^{-\varphi}dV$. Fix (s, z_0) and take extremals $f_R \in A^2(D_R)$ with $\|f_R\|_{D_R,\zeta} = 1$ and $|f_R(z_0)|^2 = K^{(R)}_\zeta(z_0,z_0)$. On each fixed D_{R_1} the weight φ is bounded above (as $\Psi \le \Phi_0$ is continuous and $\overline{D_{R_1}} \subset T$ is compact), so $\int_{D_{R_1}}|f_R|^2 dV$ is bounded uniformly in $R > R_1$, and the sub-mean value property gives locally uniform bounds for $|f_R|$. By Montel's theorem, a subsequence converges locally uniformly on T to $F \in \mathcal{O}(T)$; by Fatou, $\int_{D_{R_1}}|F|^2 e^{-\varphi}dV \le 1$ for all R_1, hence $\|F\|_{T,\zeta} \le 1$, i.e. $F \in \mathcal{H}_{\Psi_s} = \mathbb{C}\cdot 1$, and

$$|F(z_0)|^2 = \lim_{R\to\infty} K^{(R)}_\zeta(z_0,z_0) \ \le\ K^T_\zeta(z_0,z_0).$$

Combined with the reverse inequality,

$$K^{(R)}_\zeta(z,z) \ \downarrow\ K^T_\zeta(z,z) = \frac{1}{2^{-n}\int_T e^{-\Psi_s}\mu_\Omega} = 2^n e^{\,g(s)},$$

using (7) (the Bergman kernel of a one-dimensional space spanned by the constant 1). A decreasing limit of plurisubharmonic functions is plurisubharmonic (it is finite here), so $(\zeta, z) \mapsto n\log 2 + g(\operatorname{Re}\zeta)$ is plurisubharmonic on $D^{(R)}$; being a subharmonic function of ζ depending only on $\operatorname{Re}\zeta$, it is convex in $\operatorname{Re}\zeta$. Thus g is convex on $(0,\infty)$, and by (6) it is continuous at 0, hence convex on $[0,\infty)$. $\square$

Lemma 6.5 (Well estimate)**.** *There is a constant C such that $g(s) \le -(\tau - n)s + C$ for all $s \ge 0$.*

Proof. By Lemma 6.2(c), $e^{-\Psi_s} \ge e^{s\tau - M_0}$ on $B_s = B(p, r_0 e^{-s/2})$, so

$$\int_T e^{-\Psi_s} \mu_\Omega \;\ge\; e^{s\tau - M_0}\, \mu_\Omega(B_s).$$

On $B(p, r_0)$ with $r_0 \le \frac{1}{2}$ we have $|z_j| \le \frac{3}{2}$, so by (1) $\mu_\Omega \ge c_0\, dV$ with $c_0 := 2^n (3/2)^{-2n}$. Since B_s is a ball of real dimension $2n$ and radius $r_0 e^{-s/2}$,

$$\mu_\Omega(B_s) \;\ge\; c_0\, c_{2n}\, r_0^{2n} e^{-ns},$$

c_{2n} being the volume of the unit ball in $\mathbb{R}^{2n}$. Hence $g(s) \le -s\tau + ns + M_0 - \log(c_0 c_{2n} r_0^{2n})$. $\square$

Proposition 6.6. *Assume $\mathcal{H}_{\Phi_0} = \mathbb{C} \cdot 1$. Then $\displaystyle\int_T \lambda^*\, d\beta \;\le\; n$.*

Proof. By Theorem 6.4, the difference quotient $q(s) := \frac{g(s)-g(0)}{s}$ is nondecreasing on $(0, \infty)$. By Lemma 6.5, $q(s) \le -(\tau - n) + \frac{C - g(0)}{s}$, so $\limsup_{s\to\infty} q(s) \le -(\tau - n)$, and by monotonicity

$$q(s) \;\le\; -(\tau - n) \qquad \text{for every } s > 0. \tag{8}$$

On the other hand, (5) gives $e^{-\Psi_s} \le e^{-\Phi_0} e^{s(\tau - \lambda^*)}$, hence, using (4),

$$\int_T e^{-\Psi_s} \mu_\Omega \;\le\; (2\pi)^n V \int_T e^{s(\tau-\lambda^*)} d\beta, \qquad \text{i.e.} \qquad g(s) - g(0) \;\ge\; -\log \int_T e^{s(\tau - \lambda^*)} d\beta.$$

Together with (8) this yields

$$\int_T e^{s(\tau-\lambda^*)}\, d\beta \;\ge\; e^{s(\tau - n)} \qquad (s > 0).$$

Put $X := \tau - \lambda^*$, so $0 \le X \le \tau$. Since $|e^{sX} - 1 - sX| \le \frac{s^2}{2} \tau^2 e^{s\tau}$ pointwise, we get $\int e^{sX} d\beta = 1 + s \int X\, d\beta + O(s^2)$ as $s \downarrow 0$, while $e^{s(\tau - n)} = 1 + s(\tau - n) + O(s^2)$. Subtracting, dividing by s and letting $s \downarrow 0$ gives $\int X\, d\beta \ge \tau - n$, i.e. $\int_T \lambda^*\, d\beta \le n$. $\square$

7 The equilibrium mass bound

Theorem 7.1. *For every $\lambda > 0$,*

$$\mathfrak{m}\big(\{z \in T :\; P_\lambda(z) < \Phi_0(z)\}\big) \;\le\; \frac{\lambda^n}{n!}, \qquad \textit{equivalently} \qquad \mathfrak{m}(C_\lambda) \;\ge\; V - \frac{\lambda^n}{n!}.$$

Only the smoothness and strict plurisubharmonicity of Φ_0 are used; the Kähler–Einstein equation plays no role in this section. Besides Lemmas 2.2 and 2.3, we use the following classical facts: the Bedford–Taylor *comparison principle* [1, 10]: if $u, v \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$ on a bounded open Ω and $\{u < v\} \Subset \Omega$, then $\int_{\{u<v\}} (i\partial\bar\partial v)^n \le \int_{\{u<v\}} (i\partial\bar\partial u)^n$; *decreasing convergence* [1]; *plurifine locality* $\mathbf{1}_{\{u>v\}} (i\partial\bar\partial \max(u, v))^n = \mathbf{1}_{\{u>v\}} (i\partial\bar\partial u)^n$ [2, 9]; the solvability of the homogeneous Dirichlet problem on a ball with continuous boundary data (Perron–Bremermann–Walsh: there is $u \in \mathrm{PSH}(B) \cap C(\overline{B})$ with $u|_{\partial B} = h$ and $(i\partial\bar\partial u)^n = 0$, namely the upper envelope of the subsolutions) [12, 1]; the standard gluing lemma for plurisubharmonic functions; and the Green function $g_A(\cdot, a)$ of a plane annulus A [11].

7.1 Bounded envelopes and their Green bounds

Fix $\lambda > 0$ and $R > 0$ so large that $\overline{B}(p, r_0) \Subset D_R = A_R^n$ (notation of Step 2 above). Put

$$u := P_\lambda^{D_R}, \qquad \sigma_R(z) := 2 \max_{1 \le j \le n} g_{A_R}(z_j, 1) \in \mathrm{PSH}(D_R), \qquad \sigma_R \le 0.$$

Lemma 7.2. *Let $C_R := \max_{t \in \partial A_R} \log |t-1|$, $M_R := \sup_{D_R} \Phi_0$, and $C' := \lambda(\log n + 2C_R) - \inf_{D_R} \Phi_0 < \infty$.*

(a) $\nu_p(\sigma_R) = 1$ and $\sigma_R(z) \ge \log \frac{|z-p|^2}{n} - 2C_R$ on D_R.

(b) For every $\varepsilon > 0$ there is a compact $D' \subset D_R$ with $\sigma_R > -\varepsilon$ on $D_R \setminus D'$.

(c) $\Phi_0 + \lambda\sigma_R \in \mathcal{F}_\lambda(D_R)$; consequently

$$\Phi_0 + \lambda\sigma_R \;\le\; u \;\le\; \Phi_0 \quad \text{on } D_R, \qquad u(z) \;\ge\; \lambda \log |z-p|^2 - C'. \tag{9}$$

In particular $u \in \mathcal{F}_\lambda(D_R)$, $u \in L^\infty_{\mathrm{loc}}(D_R \setminus \{p\})$, and for $j > 0$

$$\{u \le \Phi_0 - j\} \;\subseteq\; \overline{B}(p, r_j), \qquad \log r_j^2 = \frac{M_R + C' - j}{\lambda},$$

so that $r_j \downarrow 0$ as $j \to \infty$.

Proof. (a) Since $g_{A_R}(\cdot, 1) - \log|\cdot - 1|$ extends harmonically across 1 and is bounded on A_R, near p we have $\sigma_R(z) = \log\big(\max_j |z_j - 1|^2\big) + O(1)$; as $\frac{1}{n}|z-p|^2 \le \max_j |z_j - 1|^2 \le |z-p|^2$, this gives $\nu_p(\sigma_R) = 1$. Globally, $h := g_{A_R}(\cdot, 1) - \log|\cdot - 1|$ is harmonic on A_R and equals $-\log|t-1| \ge -C_R$ on ∂A_R, so $h \ge -C_R$ on A_R by the minimum principle; hence $\sigma_R \ge \log\big(\max_j |z_j - 1|^2\big) - 2C_R \ge \log \frac{|z-p|^2}{n} - 2C_R$.

(b) Given $\varepsilon > 0$, by continuity of $g_{A_R}(\cdot, 1)$ up to ∂A_R and its vanishing there, there is a compact $A' \subset A_R$ with $g_{A_R}(\cdot, 1) > -\varepsilon/2$ on $A_R \setminus A'$. If $z \in D_R \setminus (A')^n$ then some $z_j \in A_R \setminus A'$, whence $\sigma_R(z) \ge 2g_{A_R}(z_j, 1) > -\varepsilon$. Take $D' := (A')^n$.

(c) $\Phi_0 + \lambda\sigma_R$ is plurisubharmonic, $\le \Phi_0$ (as $\sigma_R \le 0$), and has Lelong number $\ge \lambda$ at p by (a) (Φ_0 is smooth). Hence it is a competitor, and $u \ge \Phi_0 + \lambda\sigma_R$; combined with (a) this gives the second inequality in (9) with the stated C'. Lemma 5.1(b) gives $\nu_p(u) \ge \lambda$, so $u \in \mathcal{F}_\lambda(D_R)$. Finally, if $u(z) \le \Phi_0(z) - j \le M_R - j$, then $\lambda \log|z-p|^2 \le M_R + C' - j$, i.e. $|z-p| \le r_j$. $\square$

7.2 Maximality off the contact set

Lemma 7.3. *$(i\partial\bar\partial u)^n = 0$ on the open set $W := \{u < \Phi_0\} \setminus \{p\} \subseteq D_R$.*

Proof. Fix $z_0 \in W$ and choose $\rho' > 0$ with $\overline{B}(z_0, \rho') \subset W$. Put

$$\delta := \min_{\overline{B}(z_0, \rho')} (\Phi_0 - u) > 0, \qquad A := \sup_{\overline{B}(z_0, \rho')} \Delta\Phi_0 < \infty, \qquad r := \min\big(\rho', \sqrt{2n\delta/A}\big),$$

and $B := B(z_0, r)$. (The minimum defining δ is attained and positive because $\Phi_0 - u$ is lower semicontinuous and positive on the compact set.)

Claim. Every $w \in \mathrm{PSH}(B)$ with $\limsup_{z \to \zeta} w(z) \le u(\zeta)$ for all $\zeta \in \partial B$ satisfies $w \le u$ on B.

First, $w \le \Phi_0 - \delta/2$ on B. Indeed, w is subharmonic on B with boundary values $\le u \le \Phi_0 - \delta$ on ∂B, so $w \le H_B[\Phi_0] - \delta$, where $H_B[\Phi_0]$ is the harmonic extension of $\Phi_0|_{\partial B}$. The function

$H_B[\Phi_0]-\Phi_0-\frac{A}{4n}\big(r^2-|z-z_0|^2\big)$ is subharmonic on B (its Laplacian is $-\Delta\Phi_0+A\geq 0$, computing in $\mathbb{R}^{2n}$) with vanishing boundary values, hence nonpositive; therefore

$$H_B[\Phi_0]-\Phi_0 \;\leq\; \frac{A}{4n}\big(r^2-|z-z_0|^2\big) \;\leq\; \frac{Ar^2}{4n} \;\leq\; \frac{\delta}{2},$$

and $w\leq \Phi_0+\frac{\delta}{2}-\delta=\Phi_0-\frac{\delta}{2}$.

Now glue: let $\widehat{w}:=\max(w,u)$ on B and $\widehat{w}:=u$ on $D_R\setminus B$. By the boundary lim sup condition and upper semicontinuity of u, $\widehat{w}$ is plurisubharmonic on D_R. Moreover $\widehat{w}\leq\Phi_0$ (on B because $w\leq\Phi_0-\delta/2$ and $u\leq\Phi_0$), and $\widehat{w}=u$ near p since $p\notin\overline{B}$; hence $\nu_p(\widehat{w})=\nu_p(u)\geq\lambda$. So $\widehat{w}\in\mathcal{F}_\lambda(D_R)$ and therefore $\widehat{w}\leq P_\lambda^{D_R}=u$, i.e. $w\leq u$ on B. This proves the claim.

Since u is upper semicontinuous on the compact sphere ∂B, choose continuous $h_k\downarrow u|_{\partial B}$, and let $u_k\in\mathrm{PSH}(B)\cap C(\overline{B})$ be the Perron–Bremermann solutions with $u_k|_{\partial B}=h_k$ and $(i\partial\bar\partial u_k)^n=0$. As $u|_B$ is a competitor for the data h_k, we get $u\leq u_k$; the sequence (u_k) is nonincreasing, and its limit $\tilde{u}$ satisfies $\limsup_{z\to\zeta}\tilde{u}(z)\leq h_k(\zeta)$ for all k, hence $\leq u(\zeta)$ on ∂B. By the Claim, $\tilde{u}\leq u$, so $u_k\downarrow u$ on B. All functions involved are bounded on $\overline{B}$ (Lemma 7.2(c), $p\notin\overline{B}$), so Bedford–Taylor decreasing convergence gives $(i\partial\bar\partial u)^n=\lim_k(i\partial\bar\partial u_k)^n=0$ on B. Covering W by such balls finishes the proof. □

7.3 The comparison scheme

Proof of Theorem 7.1. Fix $\lambda>0$, $\varepsilon>0$, $\delta>0$, and R large as above. Keep the notation u,σ_R,C',M_R,r_j of Lemma 7.2 and set

$$O:=\{z\in D_R:\; u(z)<\Phi_0(z)-\varepsilon\}\;\;\text{(open)},\qquad u_j:=\max(u,\Phi_0-j)\in\mathrm{PSH}\cap L^\infty_{\mathrm{loc}}(D_R).$$

Step 0 (compact containment). By (9) and Lemma 7.2(b) (applied with ε/λ) there is a compact $D'\subset D_R$ with $u\geq\Phi_0-\varepsilon$ on $D_R\setminus D'$; hence $O\subseteq D'$, i.e. $O\Subset D_R$. Also $\{u_j<\Phi_0-\varepsilon\}=O$ for every $j>\varepsilon$.

Step 1 (global comparison). Apply the comparison principle on the bounded domain D_R to the pair $(u_j,\Phi_0-\varepsilon)$, both in $\mathrm{PSH}\cap L^\infty_{\mathrm{loc}}(D_R)$, with $\{u_j<\Phi_0-\varepsilon\}=O\Subset D_R$:

$$\int_O(i\partial\bar\partial\Phi_0)^n \;\leq\; \int_O(i\partial\bar\partial u_j)^n. \tag{10}$$

Step 2 (localization at p). Split $O=\big(O\cap\{u>\Phi_0-j\}\big)\cup\big(O\cap\{u\leq\Phi_0-j\}\big)$. On $D_R\setminus\{p\}$ both u and Φ_0-j are locally bounded, so plurifine locality gives $\mathbf{1}_{\{u>\Phi_0-j\}}(i\partial\bar\partial u_j)^n=\mathbf{1}_{\{u>\Phi_0-j\}}(i\partial\bar\partial u)^n$ there; since $O\setminus\{p\}\subseteq W$, Lemma 7.3 makes this contribution vanish. (Note $p\notin\{u>\Phi_0-j\}$, because $u(p)=-\infty$ as $\nu_p(u)\geq\lambda>0$.) By Lemma 7.2(c), $O\cap\{u\leq\Phi_0-j\}\subseteq\overline{B}(p,r_j)$. Hence

$$\int_O(i\partial\bar\partial u_j)^n \;\leq\; \int_{\overline{B}(p,r_j)}(i\partial\bar\partial u_j)^n. \tag{11}$$

Step 3 (model cap). Fix $\rho_1\in(0,r_0)$ with $\overline{B}(p,\rho_1)\subset D_R$, and set

$$M_1:=\sup_{\overline{B}(p,\rho_1)}\Phi_0,\quad m_1:=\inf_{\overline{B}(p,\rho_1)}\Phi_0,\quad \tilde\lambda:=(1+\delta)\lambda,$$

$$w(z):=\tilde\lambda\log\frac{|z-p|^2}{\rho_1^2}+M_1+1,\qquad w^{(m)}:=\max(w,-m)\quad(m>0).$$

Choose $\kappa_0 > 0$ with $2\kappa_0\tilde{\lambda} < \frac{1}{2}$. On the collar $\{\rho_1 e^{-\kappa_0} < |z-p| < \rho_1\}$ we have $w^{(m)} \geq w > M_1 + \frac{1}{2} > \Phi_0 \geq u_j$, so

$$\{w^{(m)} < u_j\} \cap B(p,\rho_1) \ \subseteq\ \overline{B}(p,\rho_1 e^{-\kappa_0}) \Subset B(p,\rho_1).$$

If moreover $m > j - m_1$, then $u_j \geq \Phi_0 - j \geq m_1 - j > -m$ on $\overline{B}(p,\rho_1)$, and therefore $\{w^{(m)} < u_j\} = \{w < u_j\}$ inside $B(p,\rho_1)$. The comparison principle applied on $B(p,\rho_1)$ to the pair $(w^{(m)}, u_j)$, together with Lemma 2.3(c) (for m large the sphere carrying the mass of $(i\partial\bar{\partial}w^{(m)})^n$ lies inside $B(p,\rho_1)$, and the total mass is $\tilde{\lambda}^n(2\pi)^n$), gives

$$\int_{\{w<u_j\}} (i\partial\bar{\partial}u_j)^n \ \leq\ \int_{B(p,\rho_1)} (i\partial\bar{\partial}w^{(m)})^n \ =\ (1+\delta)^n\lambda^n(2\pi)^n. \tag{12}$$

Finally, $\overline{B}(p,r_j) \subseteq \{w < u_j\}$ for all large j: on $\overline{B}(p,r_j)$ (which is contained in $B(p,\rho_1)$ for large j) we have $u_j \geq m_1 - j$, while by the formula for r_j,

$$w \ \leq\ \tilde{\lambda}\log\frac{r_j^2}{\rho_1^2} + M_1 + 1 \ =\ (1+\delta)\big(M_R + C' - j\big) + c_1$$

with c_1 independent of j; hence $w - u_j \leq -\delta j + O(1) < 0$ for $j \geq j_0(\delta)$.

Choosing such a j and then m large, and combining (10)–(12) and dividing by $(2\pi)^n n!$:

$$\mathfrak{m}\big(\{P_\lambda^{D_R} < \Phi_0 - \varepsilon\}\big) \ \leq\ \frac{(1+\delta)^n\lambda^n}{n!} \qquad \text{for all } R, \varepsilon, \delta > 0. \tag{13}$$

Step 4 (limits). If $R < R'$, restriction maps $\mathcal{F}_\lambda(D_{R'})$ into $\mathcal{F}_\lambda(D_R)$, so $P_\lambda^{D_{R'}} \leq P_\lambda^{D_R}$ on D_R; moreover every global competitor restricts, so $P_\lambda^{D_R} \geq P_\lambda$. Hence $Q := \lim_{R\to\infty} P_\lambda^{D_R}$ exists, is upper semicontinuous, satisfies $P_\lambda \leq Q \leq \Phi_0$, and (by the uniform bound of Lemma 5.1(b), whose constants r_0, M_0 do not depend on R) obeys $Q \leq \lambda\log\frac{|z-p|^2}{r_0^2} + M_0$ on $B(p,r_0)$, so $\nu_p(Q) \geq \lambda$. If $Q \not\equiv -\infty$ then Q is plurisubharmonic on T, hence $Q \in \mathcal{F}_\lambda(T)$ and $Q \leq P_\lambda$; thus $Q = P_\lambda$. If $Q \equiv -\infty$ then also $P_\lambda \equiv -\infty$ and again $Q = P_\lambda$. Consequently

$$\{P_\lambda^{D_R} < \Phi_0 - \varepsilon\} \ \uparrow\ \{P_\lambda < \Phi_0 - \varepsilon\} \qquad (R \uparrow \infty),$$

an increasing union. Letting $R \uparrow \infty$ in (13) (monotone convergence), then $\varepsilon \downarrow 0$, then $\delta \downarrow 0$, we obtain

$$\mathfrak{m}\big(\{P_\lambda < \Phi_0\}\big) \ \leq\ \frac{\lambda^n}{n!}. \qquad \square$$

Remark 7.4. No statement about conservation of mass at infinity is needed: the Green collar of Lemma 7.2(b) compactifies each stage. Note also that the stagewise envelope $P_\lambda^{D_R}$ is locally bounded off p, with a model-type lower bound (9), even though the global envelope P_λ may degenerate; the scheme is insensitive to this, since $\sigma_R \downarrow -\infty$ pointwise as $R \to \infty$.

Remark 7.5. Taking $\lambda < \tau$ in Theorem 7.1 gives $\mathfrak{m}(C_\lambda) \geq V - \lambda^n/n! > 0$; in particular $C_\lambda \neq \emptyset$ and $P_\lambda \not\equiv -\infty$. Thus the nondegeneracy of the envelopes is a corollary of the argument rather than an input.

8 Proof of Theorem 1.2

Let K be as in Theorem 1.2, let ϕ be the potential of §3, $\Phi_0 = \phi \circ L$, $\mathfrak{m} = \mathfrak{m}_{\Phi_0}$ and $\beta = \mathfrak{m}/V$; the identity (4) is the only place where hypothesis (i) ($b(K) = 0$) is used. By Lemma 4.2, $\mathcal{H}_{\Phi_0} = \mathbb{C} \cdot 1$; this is the only place where hypothesis (ii) ($\mathrm{int} K \cap \mathbb{Z}^n = \{0\}$) is used.

By Lemma 5.3 applied with $\mu = \mathfrak{m}$, then Theorem 7.1 (note $V - \lambda^n/n! \geq 0$ for $\lambda \leq \tau$),

$$\int_T \lambda^* \, d\mathfrak{m} \;=\; \int_0^\tau \mathfrak{m}(C_\lambda)\, d\lambda \;\geq\; \int_0^\tau \Big(V - \frac{\lambda^n}{n!}\Big) d\lambda \;=\; V\tau - \frac{\tau^{n+1}}{(n+1)!} \;=\; \frac{n}{n+1}\,\tau V,$$

the last equality because $\tau^n = n!V$, so $\frac{\tau^{n+1}}{(n+1)!} = \frac{\tau V}{n+1}$.

On the other hand, Proposition 6.6 and $\mathfrak{m} = V\beta$ give

$$\int_T \lambda^* \, d\mathfrak{m} \;=\; V \int_T \lambda^* \, d\beta \;\leq\; nV.$$

Combining, $\frac{n}{n+1}\tau V \leq nV$, i.e. $\tau \leq n+1$, i.e.

$$n! \operatorname{vol}(K) \;\leq\; (n+1)^n. \qquad \blacksquare$$

This proves Theorem 1.2, and hence Theorem 1.1.

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