An unconditional chain, not asserted.

Abstract

A from–scratch solver, run with no hints and no access to prior work on the problem (audited in full), was asked to resolve the Erdős–Szemerédi sum–product conjecture. It found the following construction, ran its own adversarial verification, assembled a chain of classical inputs that would make the construction unconditional, marked that chain [UNVERIFIED], and then declined to assert the result on the grounds that the conclusion "contradict[s] the expert consensus" and is "far more likely to indicate an undetected error than to be true". This document records the chain in the form of estimates (A)–(F) (§1), the solver's own verification (§2), and its stated reason for not asserting the result, verbatim (§3).

1 The chain.

1.1 Setup.

Let K be a totally real number field of degree d with ring of integers $\mathfrak{o}_K$, discriminant Δ_K, regulator Reg_K, and root discriminant $|\Delta_K|^{1/d} =: r \le r_0$. Fix one real embedding $K \hookrightarrow \mathbb{R}$; distinct elements of K give distinct reals. For $C \ge 1$ set

$$U_K(C) := \left\{u \in \mathfrak{o}_K^\times : \ |\sigma(u)| \in [C^{-1}, C] \text{ for every archimedean } \sigma\right\};$$

write $m := \#|U_K(C_0)|$. Fix constants $C_0 \ge 2$, $K_0 \ge 10^3$, $R := K_0 C_0^2 r$. Choose $x_1, \dots, x_s \in \mathfrak{o}_K \setminus \{0\}$ of house $\le R/C_0$, pairwise inequivalent under multiplication by $U_K(C_0^2)$, and set

$$B := \bigcup_{j \le s} U_K(C_0) \cdot x_j \subset \mathfrak{o}_K \subset \mathbb{R}, \qquad n := \#|B| = ms.$$

The orbits are disjoint and each has exactly m elements.

1.2 The hypothesis.

Hypothesis 1.1 (U′). *There exist absolute constants $C_0 \geq 2$, $\varepsilon > 0$, $r_0 \geq 1$, and totally real K of arbitrarily large degree d with $|\Delta_K|^{1/d} \leq r_0$, such that*

$$\#|U_K(C_0)| \;\geq\; (5+\varepsilon)^d.$$

Equivalently, by Dirichlet's unit theorem and van der Corput on the log–unit lattice, (U′) asserts the existence of totally real fields of unbounded degree with bounded root discriminant and $\mathrm{Reg}_K \leq O_{r_0}(1)^d$. Zimmert's bound $\mathrm{Reg}_K \geq 0.04\, e^{0.46d}$ [KNOWN] shows regulators grow at least exponentially but does not contradict (U′) for large C_0.

1.3 The estimates.

The solver established the following (its tags [SKETCH] and [KNOWN] retained).

(A) $\#|\{x \in \mathfrak{o}_K : \mathrm{house}(x) \leq M\}| \leq (2M+1)^d$ (distinct algebraic integers are 1–separated in ℓ^∞). [SKETCH; VERIFIED]

(B) $\#|\{x \in \mathfrak{o}_K : \mathrm{house}(x) \leq M\}| \geq M^d/r^{d/2}$ once $M \gg_{r_0} 1$. [KNOWN; VAN DER CORPUT]

(C) $\#|U_K(C^2)| \leq 5^{d-1}\, \#|U_K(C)|$. [SKETCH; LATTICE COVERING]

(D) $s \geq (R/C_0)^d/(5^{d-1} m\, r^{d/2})$ (greedy orbit selection). [SKETCH]

(E) $\#|B \cdot B| \leq \#|U_K(C_0^2)| \cdot \binom{s+1}{2} \leq 5^{d-1} m s^2$. [SKETCH; VERIFIED]

(F) $\#|B+B| \leq (4R+1)^d$ (every element has house $\leq 2R$). [SKETCH; VERIFIED]

And, independently of (A)–(F), the chain for Hypothesis 1.1 itself:

(C′) Totally real towers of bounded root discriminant exist (Martinet/Golod–Shafarevich). [KNOWN]

(C″) Along any such tower, $\mathrm{Reg}_K \leq O_{r_0}(1)^d$: the class number formula $h_K\,\mathrm{Reg}_K = 2^{1-d}|\Delta_K|^{1/2}\,\mathrm{Res}_{s=1}\,\zeta_K$, the bound $h_K \geq 1$, and an unconditional Rademacher/Louboutin–type bound $\mathrm{Res}_{s=1}\,\zeta_K \leq O_{r_0}(1)^d$. [UNVERIFIED CHAIN][1]

(C‴) $\#|U_K(C_0)| \;\geq\; (\log C_0)^{d-1} \cdot O_{r_0}(1)^{-d}/\,\mathrm{Reg}_K$ (van der Corput on the log–unit lattice), so (C″) ⇒ (U′) with $C_0 = O_{r_0}(1)$.

[1] The label [UNVERIFIED CHAIN] is the solver's own marking for this argument, not ours; see §2.

1.4 The tally.

Assuming (A)–(F) and (U′): $n = ms \geq (R/C_0)^d/(5^{d-1}r^{d/2})$ by (D), so

$$\frac{\#|B+B|}{n^2} \leq \Big(\frac{(4R+1)\cdot 25C_0^2 r}{R^2}\Big)^d \approx \Big(\frac{100}{K_0}\Big)^d, \quad \frac{\#|B\cdot B|}{n^2} \leq \frac{5^{d-1}}{m} \leq \Big(\frac{5}{5+\varepsilon}\Big)^d,$$

and since $n \leq (2R+1)^d$ both are $\leq n^{-\S}$ with

$$\S \;=\; \frac{\min\big(\log(K_0/100),\ \log(1+\varepsilon/5)\big)}{\log(2K_0C_0^2 r_0+1)} \;>\; 0.$$

Thus (U′) + (A)–(F) $\Rightarrow$ (NEG); and (C′) + (C″) + (C‴) $\Rightarrow$ (U′). The discriminant does not improve the container bound (F); it enters only the lower bound (B) for n, which is why bounded root discriminant must be hypothesized.

2 The solver's verification.

The solver commissioned an independent adversarial re–derivation (attacking each lemma, re–deriving the bookkeeping, and checking consistency with the literature). Findings, in its words:

Two genuine defects, both repairable. (i) The originally proposed constant $R = 100C_0^2 r$ makes (F) fail to beat n^2 for large d; $K_0 \geq 10^3$ repairs it. (ii) Bounded root discriminant is not automatic from the presence of many small units; it must be part of the hypothesis, as in (U′).

No refutation of the repaired implication. After the two repairs, the container count (A), covering constant (C), greedy selection (D), distinctness, and the bookkeeping of §1.4 all survived independent re–derivation. Consistency checks: no known theorem (Chang's bounded–degree sum–product, Evertse–Schlickewei–Schmidt unit–equation counting — whose bounds are exponential in the rank and hence vacuous at rank $\asymp \log n$ —, divisor bounds, Zimmert/Friedman regulator lower bounds) contradicts the construction.

The destabilizing finding. The verifier's own words:

> So the construction does not contradict any known theorem in additive combinatorics — but the *package* contradicts itself: if Steps 1–2 are right, then together with (e) it yields sets with $\max(|B+B|, |B\cdot B|) \leq n^{2-\delta}$, $\delta \approx 2\times 10^{-5}$, i.e. an **unconditional disproof of ES**. That conclusion is extraordinary; the overwhelmingly likely resolution is that there is an error either in Steps 1–2 (which I could not find beyond the repairable items above) or in the author's/ my assessment of the

analytic input. The author's stated reason for openness ("loss of a log per character", i.e. $\rho_K \le (c \log d_K)^{d-1} = e^{O(d \log d)}$) ignores that the conductors multiply to d_K (AM–GM in the abelian case) and that the convexity bound achieves the same $\left(\frac{\log d_K}{d}\right)^d$ shape in general; this is the single point on which "safely conditional" versus "unconditional and unbelievable" turns, and it must be checked against the literature (Louboutin, *J. Number Theory/Canad. J. Math.* 2000–01; Rademacher 1959) before the claim is circulated.

(ii) Hypothesis (U): NOT OPEN as claimed; modulo a literature check it is KNOWN TRUE — and that is fatal for the narrative, not a victory. For $C_0 \le 2$ it is known false (Kronecker); for large absolute C_0, the form actually needed (with bounded root discriminant) follows from Martinet's totally real towers + the class number formula + $h \ge 1$ + the classical residue/convexity bound + the author's own Lemma 1. Consequently the document, as corrected by its own tools, is not a conditional reduction but an implicit unconditional disproof of Erdős–Szemerédi. Given the strong consensus that ES is true, the rational conclusion is that the route contains an as-yet-unlocated error — the prime suspects being either a subtle flaw in Steps 1–2 that has survived both the author's and my scrutiny, or a misassessment (mine or the author's) of the residue bound. Until this tension is resolved, the claimed theorem should not be accepted.

3 Why the solver does not assert the result.

Verbatim from the solver's final output (§2.4):

> Given the strength of the consensus for the conjecture and the fact that the construction uses only elementary counting on top of classical inputs, the overwhelmingly likely explanation is an error that neither pass located — the prime suspects being (a) a subtle flaw in the combinatorial steps of §2.2 (e.g. in the interaction between the greedy selection and the two container bounds), (b) a misquotation or misapplication of the residue bound or of the existence of totally real towers of bounded root discriminant, or (c) an incompatibility between the parameter regimes of the cited results that disappears only when all constants are written out in full. Deciding which requires a level of expert scrutiny (and ideally a numerical experiment on a moderate–degree field with small regulator) that is beyond what this investigation can certify.

And its recommended next step (§5):

the first task is to verify the exact statement and applicability of the Rademacher/Louboutin residue bound and of Martinet–type totally real towers; the second is a full constant–tracked write–up refereed by a number theorist; the third is a numerical study of unit counts and of $|B+B|, |B \cdot B|$ in moderate–degree fields of small regulator.

4 The solver's arguments for each estimate.

We collect here, estimate by estimate, the arguments that the solver itself produced; the standing setup is that of §1.1. The labels (A)–(F) and (C′)–(C‴) are this writeup's reorganization of what the solver called Lemmas 1–4 and surrounding prose; the arguments, and the tag [UNVERIFIED CHAIN] on (C″), are the solver's own.

4.1 (A) The additive container: upper bound.

$$\#\bigl|\{x \in \mathfrak{o}_K : \mathrm{house}(x) \le M\}\bigr| \;\le\; (2M+1)^d.$$

Take distinct $x, y \in \mathfrak{o}_K$. Then $x - y \neq 0$, so its norm is a nonzero rational integer, and

$$\prod_i \bigl|x^{(i)} - y^{(i)}\bigr| \;=\; \bigl|\mathrm{N}_{K/\mathbb{Q}}(x-y)\bigr| \;\ge\; 1,$$

which forces $\max_i |x^{(i)} - y^{(i)}| \ge 1$. Thus the Minkowski images of distinct integers are 1–separated in ℓ^∞, and the box $[-M, M]^d$ can contain at most $(2M+1)^d$ of them. The solver stresses the asymmetry: the discriminant does *not* enter this upper bound; it enters the true count downward.

4.2 (B) The additive container: lower bound.

$$\#\bigl|\{x \in \mathfrak{o}_K : \mathrm{house}(x) \le M\}\bigr| \;\ge\; M^d/r^{d/2} \qquad (M \gg_{r_0} 1).$$

Here the solver records only the invocation: writing

$$P \;:=\; \#\bigl|\{x \in \mathfrak{o}_K : \mathrm{house}(x) \le R/C_0\}\bigr|,$$

van der Corput on the lattice $\mathfrak{o}_K$ (covolume $r^{d/2}$) gives

$$P \;\ge\; (R/C_0)^d \big/ r^{d/2}.$$

No further argument is supplied.

4.3 (C) Lattice doubling of U_K.

$$\#\big|U_K(C^2)\big| \;\le\; 5^{\,d-1}\,\#\big|U_K(C)\big|.$$

The solver states the general fact first: for a lattice Λ and a symmetric convex body Q in a k–dimensional real vector space,

$$\#\big|\Lambda\cap 2Q\big| \;\le\; 5^k\,\#\big|\Lambda\cap Q\big|.$$

To see this, pick a maximal set $\{z_i\}\subseteq\Lambda\cap 2Q$ whose pairwise differences avoid $\operatorname{int}Q$. The bodies $z_i+\frac{1}{2}Q$ are then pairwise disjoint and all lie inside $\frac{5}{2}Q$, so a volume comparison bounds their number by 5^k. Maximality forces every $v\in\Lambda\cap 2Q$ to satisfy $v-z_i\in\Lambda\cap Q$ for some i, i.e.

$$\Lambda\cap 2Q \;\subseteq\; \bigcup_i\big(z_i+(\Lambda\cap Q)\big),$$

which gives the count. Applying this with $k=d-1$ (the log–unit lattice inside the trace–zero hyperplane), $Q_{C_0^2}=2Q_{C_0}$, and noting that the torsion ±1 appears identically on both sides, one obtains

$$\#\big|U_K(C_0^2)\big| \;\le\; 5^{\,d-1}\,\#\big|U_K(C_0)\big|.$$

4.4 (D) Greedy orbit selection.

$$s \;\ge\; \frac{(R/C_0)^d}{5^{\,d-1}\,m\,r^{d/2}}.$$

The solver's procedure is greedy: choose $x_1,\dots,x_s\in\mathfrak{o}_K$ of house $\le R/C_0$ with the property that $x_i/x_j\notin U_K(C_0^2)$ for $i\ne j$. By (B) there are at least $(R/C_0)^d/r^{d/2}$ integers of house $\le R/C_0$ to draw from, and each chosen x_i rules out at most $\#|U_K(C_0^2)|$ further candidates; with (C) this yields

$$s \;\ge\; \frac{(R/C_0)^d}{r^{d/2}\cdot 5^{\,d-1}\,m}.$$

For distinctness of $B:=\bigcup_{j\le s}U_K(C_0)\,x_j$: if $ux_i=u'x_j$ with $u,u'\in U_K(C_0)$, then

$$x_i/x_j \;=\; u'/u \;\in\; U_K(C_0)\cdot U_K(C_0)^{-1} \;\subseteq\; U_K(C_0^2),$$

which the selection excludes unless $i=j$, and then $u=u'$. Hence the orbits are disjoint and $\#|B|=ms$ exactly.

4.5 (E) Product set bound.

$$\#\big|B\cdot B\big| \;\le\; \#\big|U_K(C_0^2)\big|\cdot\binom{s+1}{2} \;\le\; 5^{\,d-1}\,m\,s^2.$$

From $U_K(C_0)\cdot U_K(C_0)\subseteq U_K(C_0^2)$ we get

$$B\cdot B \;\subseteq\; \bigcup_{i\le j} U_K(C_0^2)\cdot x_ix_j,$$

so the product set is covered by $\binom{s+1}{2}$ unit–orbits of size $\#|U_K(C_0^2)|$, whence

$$\#\big|B{\cdot}B\big| \;\le\; \binom{s+1}{2}{\cdot}\#\big|U_K(C_0^2)\big| \;\le\; \binom{s+1}{2}{\cdot}5^{\,d-1}m \;=\; 5^{\,d-1}\,\frac{n^2}{m}\,(1{+}o(1)).$$

4.6 (F) Sum set bound.

$$\#\big|B+B\big| \;\le\; (4R+1)^d.$$

Each element of B has house $\le C_0\cdot(R/C_0)=R$, so every element of $B+B$ has house $\le 2R$; applying (A) with $M=2R$ gives the display.

The solver then reworks the bookkeeping and flags a repair. With the originally proposed $R=100\,C_0^2r$, combining (D) with $n=ms$ gives

$$n \;\ge\; 5\,(20\,C_0\sqrt{r})^d, \qquad \#|B+B| \;\le\; \big((20\,C_0\sqrt{r})^2+1\big)^d,$$

so that

$$\frac{\#|B+B|}{n^2} \;\le\; \frac{\big(1+\frac{1}{400\,C_0^2r}\big)^d}{25},$$

which is never a power saving and in fact exceeds 1 once $d\gg C_0^2r$. The remedy is to take $R=K\,C_0^2r$ with $K\ge 10^3$; then

$$\#|B+B| \;\le\; n^2\cdot(O(1)/K)^d,$$

a genuine power saving as long as $r\le r_0$ stays bounded.

4.7 (C′), (C″), (C‴): the chain for Hypothesis (U′).

(C′) *Totally real towers of bounded root discriminant exist.*

The solver invokes Martinet's infinite totally real class–field tower (Golod–Shafarevich), whose root discriminant is an absolute $r_0=O(1)$; the recorded value is $r_0\approx 913$.

(C″) *Along any such tower,* $\mathrm{Reg}_K \le O_{r_0}(1)^d$. [UNVERIFIED CHAIN]

The chain runs

$$\mathrm{Reg}_K \;\le\; h_K \,\mathrm{Reg}_K \;=\; 2^{1-d}\,|\Delta_K|^{1/2}\,\mathrm{Res}_{s=1}\,\zeta_K(s),$$

using $h_K \ge 1$ together with the unconditional convexity bound (Rademacher; in the explicit form attributed to Louboutin)

$$\mathrm{Res}_{s=1}\,\zeta_K(s) \;\le\; \left(\frac{e\,\log|\Delta_K|}{2(d-1)}\right)^{d-1} \;\le\; O_{r_0}(1)^d$$

valid whenever $|\Delta_K|^{1/d} \le r_0$. Putting these together,

$$\mathrm{Reg}_K \;\le\; 2^{1-d}\,r_0^{d/2}\cdot O_{r_0}(1)^d \;=\; O_{r_0}(1)^d.$$

At $r_0 \approx 913$ the solver's numerical instantiation reads $\mathrm{Reg}_K \le e^{6d}$. This is the step the solver itself tagged [UNVERIFIED CHAIN].

(C‴) $\#|U_K(C_0)| \ge (\log C_0)^{d-1}\cdot O_{r_0}(1)^{-d}/\,\mathrm{Reg}_K$, *so* (C″) $\Rightarrow$ (U′).

Van der Corput applied to the log–unit lattice (covolume $\sqrt{d}\cdot\mathrm{Reg}_K$ in the trace–zero hyperplane) gives

$$\#\big|U_K(C_0)\big| \;\ge\; \frac{c\,(\log C_0)^{d-1}}{\sqrt{d}\;\mathrm{Reg}_K}.$$

Feeding in (C″) turns this into $\#|U_K(C_0)| \ge O_{r_0}(1)^{-d}(\log C_0)^{d-1}$; choosing $C_0 = O_{r_0}(1)$ large enough (with $\log C_0 \approx 2000$ at $r_0 \approx 913$) then forces

$$\#\big|U_K(C_0)\big| \;\ge\; (5+\varepsilon)^d$$

for an absolute $\varepsilon > 0$, which is Hypothesis (U′).

Remark 4.1. Per the solver's own tags reproduced in §1.3: (A), (E), (F) are marked [SKETCH; VERIFIED]; (C), (D) are marked [SKETCH]; (B) and (C′) are marked [KNOWN]; (C″) is marked [UNVERIFIED CHAIN].