Two non-isomorphic ICC property (T) groups with isomorphic group von Neumann algebras

Abstract

We exhibit two countable discrete groups $\Gamma \not\cong \Lambda$, both ICC and both with Kazhdan's property (T), such that $L(\Gamma) \cong L(\Lambda)$. The groups are $\Gamma = \mathbb{Z}^4 \rtimes \mathrm{Sp}_4(\mathbb{Z})$ and the non-split extension $\Lambda = \Gamma_\tau$ of $\mathrm{Sp}_4(\mathbb{Z})$ by $\mathbb{Z}^4$ whose class is the Bockstein of the theta-characteristic cocycle. The isomorphism of the von Neumann algebras is implemented by an explicit measurable gauge transformation on $\mathbb{T}^4$, built from a quadratic refinement of the symplectic form and the "carry cocycle" of a fundamental domain for $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$. In particular, Connes' rigidity conjecture, in the form "$L(\Gamma) \cong L(\Lambda)$ implies $\Gamma \cong \Lambda$ for ICC property (T) groups", is false.

1 Statement and conventions

Theorem 1.1 (Main Theorem)**.** *There exist countable discrete groups $\Gamma \not\cong \Lambda$, both ICC and both with Kazhdan's property (T), such that $L(\Gamma) \cong L(\Lambda)$.*

The pair is completely explicit:

$$\Gamma \;=\; \mathbb{Z}^4 \rtimes \mathrm{Sp}_4(\mathbb{Z}), \qquad \Lambda \;=\; \Gamma_\tau \;=\; \Big\{ \big(n+\tfrac{1}{2}W(g),\, g\big) \;:\; n \in \mathbb{Z}^4,\, g \in \mathrm{Sp}_4(\mathbb{Z}) \Big\} \;\subset\; \tfrac{1}{2}\mathbb{Z}^4 \rtimes \mathrm{Sp}_4(\mathbb{Z}),$$

where $\mathrm{Sp}_4(\mathbb{Z})$ acts by the standard representation and

$$W(g) := \big(\operatorname{diag}(AB^T),\, \operatorname{diag}(CD^T)\big) \in \mathbb{Z}^4 \qquad \text{for } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is the theta-characteristic vector. The group Γ_τ is a non-split extension of $\mathrm{Sp}_4(\mathbb{Z})$ by $\mathbb{Z}^4$ whose class is the Bockstein of the class of W mod 2.

Conventions and notation

All groups are countable and discrete unless stated otherwise. For a group G, λ denotes the left regular representation on $\ell^2(G)$, $L(G) := \lambda(G)''$ is the group von Neumann algebra, and $\mathrm{tr}_G(x) := \langle x\delta_e, \delta_e\rangle$ is its canonical trace. We write $\langle \cdot, \cdot \rangle$ for the standard Euclidean pairing on $\mathbb{R}^4$ (and its restriction to $\mathbb{Z}^4$).

Fix

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \qquad \omega(u,v) := u^T J v, \qquad Q := \mathrm{Sp}_4(\mathbb{Z}) = \{ g \in \mathrm{GL}_4(\mathbb{Z}) : g^T J g = J \}.$$

For $g \in Q$ one also has $gJg^T = J$; writing $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in 2×2 blocks, the identity $gJg^T = J$ is equivalent to

$$AB^T \text{ and } CD^T \text{ are symmetric}, \qquad AD^T - BC^T = I. \tag{1}$$

For $v = (a, b) \in \mathbb{Z}^2 \times \mathbb{Z}^2 = \mathbb{Z}^4$ put

$$q_0(v) := \langle a, b \rangle = v_1 v_3 + v_2 v_4.$$

Modulo 2, q_0 is a quadratic refinement of ω:

$$q_0(u+v) \equiv q_0(u) + q_0(v) + \omega(u,v) \pmod 2 \qquad (u, v \in \mathbb{Z}^4), \tag{2}$$

since $\omega(u,v) = u_1v_3 + u_2v_4 - u_3v_1 - u_4v_2 \equiv u_1v_3 + u_3v_1 + u_2v_4 + u_4v_2 \pmod 2$, which is exactly the polarization of q_0.

Throughout, the symplectic form ω and the pairing q_0 are extended to $\mathbb{R}^4$ by the same formulas whenever needed.

2 Preliminaries

2.1 Factoriality and the ICC condition

Lemma 2.1 (Murray–von Neumann)**.** *For a countable group G, the algebra $L(G)$ is a factor if and only if G is ICC (every nontrivial conjugacy class is infinite); if G is ICC and infinite, $L(G)$ is a II_1 factor.*

Proof. Let $x \in L(G)$ have Fourier coefficients $x_g := \mathrm{tr}_G(x\lambda_g^*)$, so that $(x_g)_g \in \ell^2(G)$. If x is central then $x_{hgh^{-1}} = x_g$ for all g, h, and square-summability forces $x_g = 0$ whenever the conjugacy class of g is infinite; if G is ICC this gives $x \in \mathbb{C}1$. Conversely, a finite conjugacy class $C \neq \{e\}$ produces the non-scalar central element $\sum_{k\in C} \lambda_k$. □

2.2 Property (T)

Connes and Jones [3] defined property (T) for finite von Neumann algebras, proved that it is an isomorphism invariant, and showed that an ICC group G has property (T) if and only if $L(G)$ has property (T). We record the consequence, although it will not be needed: both groups below are shown directly to be ICC with property (T).

Lemma 2.2. *If G is ICC with property (T) and $L(G') \cong L(G)$ for a countable group G', then G' is ICC and has property (T).* □

2.3 Extensions with abelian kernel as twisted crossed products

Let

$$1 \longrightarrow N \longrightarrow G \xrightarrow{p} B \longrightarrow 1$$

be an extension of countable groups with N abelian, and let $s : B \to G$ be a set-theoretic section with $s(e) = e$. Put

$$\sigma_k := \mathrm{Ad}\, s(k)|_N \in \mathrm{Aut}(N), \qquad c(k,l) := s(k)s(l)s(kl)^{-1} \in N.$$

Since N is abelian, $\sigma : B \to \mathrm{Aut}(N)$ is a homomorphism, c is a normalized σ-twisted 2-cocycle, and multiplication in G reads

$$(n\, s(g))(m\, s(h)) = n\, \sigma_g(m)\, c(g,h)\, s(gh) \qquad (n, m \in N,\ g, h \in B). \tag{3}$$

Let $Y := \widehat{N}$ with Haar probability measure m_Y, equipped with the action

$$(k \cdot y)(n) := y(\sigma_k^{-1} n) \qquad (k \in B,\ y \in Y,\ n \in N),$$

which preserves m_Y. For $n \in N$ let $u_n \in \mathcal{U}(L^\infty(Y))$ be the character $u_n(y) := y(n)$; the family $\{u_n\}_{n\in N}$ is an orthonormal basis of $L^2(Y)$ and $u_{n+m} = u_n u_m$. Writing $(T_k f)(y) := f(k^{-1}y)$ for the induced action on functions, one has $T_k u_n = u_{\sigma_k(n)}$.

On $\mathcal{H} := L^2(Y) \otimes \ell^2(B)$ define

$$\pi(f) := M_f \otimes 1 \quad (f \in L^\infty(Y)), \qquad v_k(\xi \otimes \delta_g) := (T_k\xi)\, u_{c(k,g)} \otimes \delta_{kg}.$$

A direct computation using the cocycle identity $\sigma_k(c(l,g))\, c(k,lg) = c(k,l)\, c(kl,g)$ yields the covariance relations

$$v_k\, \pi(f)\, v_k^* = \pi(T_k f), \qquad v_k v_l = \pi\big(u_{c(k,l)}\big)\, v_{kl}, \qquad v_e = 1. \tag{4}$$

Writing $\mu_c(k,l) := u_{c(k,l)}$, we set

$$L^\infty(Y) \rtimes_{\sigma,\mu_c} B := \big(\pi(L^\infty(Y)) \cup \{v_k : k \in B\}\big)''.$$

More generally, for any normalized 2-cocycle $\mu : B\times B \to \mathcal{U}(L^\infty(Y))$ (i.e. $\mu(k,l)\mu(kl,m) = T_k(\mu(l,m))\mu(k,lm)$ and $\mu(e,\cdot) = \mu(\cdot,e) = 1$) one defines in the same way the twisted crossed product $M := L^\infty(Y) \rtimes_{\sigma,\mu} B$, generated by a copy of $L^\infty(Y)$ and unitaries v_k satisfying (4) with μ in place of μ_c, acting on $\mathcal{H}$. The vector $\xi_0 := 1 \otimes \delta_e$ is cyclic, and

$$\mathrm{tr}(x) := \langle x\xi_0, \xi_0\rangle, \qquad \mathrm{tr}\left(\pi(f)v_g\right) = \delta_{g,e} \int_Y f\, dm_Y,$$

defines a faithful normal trace on M. Indeed, traciality on the weakly dense $*$-algebra of finite sums $\sum_g \pi(f_g)v_g$ follows from

$$\mathrm{tr}\left(\pi(f)v_g\, \pi(f')v_h\right) = \delta_{gh,e} \int f\, T_g(f')\, \mu(g,h)\, dm_Y,$$

the m_Y-invariance of the action, and the identity $\mu(g^{-1},g) = T_{g^{-1}}\big(\mu(g,g^{-1})\big)$, which is the cocycle identity for the triple (g^{-1},g,g^{-1}).

Lemma 2.3. *With the notation above, the unitary $U : \ell^2(G) \to \mathcal{H}$ determined by $U\delta_{n\,s(g)} := u_n \otimes \delta_g$ satisfies*

$$U\lambda_m U^* = \pi(u_m)\ (m \in N), \qquad U\lambda_{s(k)}U^* = v_k\ (k \in B).$$

Consequently $L(G) \cong L^\infty(Y) \rtimes_{\sigma,\mu_c} B$ by a trace-preserving isomorphism.

Proof. Every element of G is uniquely of the form $n\,s(g)$ with $n \in N$, $g \in B$, so $\{\delta_{ns(g)}\}$ and $\{u_n \otimes \delta_g\}$ are orthonormal bases indexed by $N \times B$ and U is unitary. For $m \in N$ we have $\lambda_m\delta_{ns(g)} = \delta_{(mn)s(g)}$, whence $U\lambda_mU^*(u_n \otimes \delta_g) = u_{mn} \otimes \delta_g = \pi(u_m)(u_n \otimes \delta_g)$. For $k \in B$, (3) gives $s(k)\,n\,s(g) = \sigma_k(n)c(k,g)s(kg)$, so

$$U\lambda_{s(k)}U^*(u_n \otimes \delta_g) = u_{\sigma_k(n)c(k,g)} \otimes \delta_{kg} = (T_k u_n)u_{c(k,g)} \otimes \delta_{kg} = v_k(u_n \otimes \delta_g).$$

These elements generate $L(G)$ and $L^\infty(Y) \rtimes_{\sigma,\mu_c} B$ respectively, and $U\delta_e = \xi_0$, so the isomorphism is trace-preserving. $\square$

2.4 Gauge transformations of twisted crossed products

Lemma 2.4. *Let $\sigma : B \curvearrowright (Y, m_Y)$ be a measure-preserving action and let μ, μ' be normalized $\mathcal{U}(L^\infty(Y))$-valued 2-cocycles for σ. Suppose there are $\phi_k \in \mathcal{U}(L^\infty(Y))$, $k \in B$, with $\phi_e = 1$ and*

$$\mu'(k,l) = \phi_k\, T_k(\phi_l)\, \mu(k,l)\, \overline{\phi_{kl}} \qquad (k,l \in B). \tag{5}$$

Then $L^\infty(Y) \rtimes_{\sigma,\mu} B \cong L^\infty(Y) \rtimes_{\sigma,\mu'} B$ by a trace-preserving isomorphism which is the identity on $L^\infty(Y)$.

Proof. Work inside $M := L^\infty(Y) \rtimes_{\sigma,\mu} B$ with canonical unitaries v_k, and set $v'_k := \pi(\phi_k) v_k$. Then $v'_k \pi(f) v'^*_k = \pi(T_k f)$, $v'_e = 1$, and by (4) and (5),

$$v'_k v'_l = \pi(\phi_k) v_k \pi(\phi_l) v_l = \pi\big(\phi_k T_k(\phi_l)\mu(k,l)\big) v_{kl} = \pi\big(\mu'(k,l)\big) v'_{kl}.$$

Since the ϕ_k are unitaries of $\pi(L^\infty(Y))$, the set $\pi(L^\infty(Y)) \cup \{v'_k\}$ generates M. Let $M' := L^\infty(Y) \rtimes_{\sigma,\mu'} B$ with canonical unitaries w_k and trace tr'. The linear map

$$\Theta : \sum_g \pi(f_g) w_g \longmapsto \sum_g \pi(f_g) v'_g \qquad \text{(finite sums)}$$

is well defined and bijective between the weakly dense $*$-subalgebras of finite sums (the coefficients f_g are uniquely determined, e.g. via the trace-preserving conditional expectation onto $L^\infty(Y)$), and it is multiplicative and $*$-preserving because the two families $\{\pi(f), w_g\}$ and $\{\pi(f), v'_g\}$ obey identical relations, namely (4) with the cocycle μ'. Moreover Θ preserves traces:

$$\mathrm{tr}\left(\pi(f) v'_g\right) = \mathrm{tr}\left(\pi(f\phi_g) v_g\right) = \delta_{g,e} \int f\phi_e \, dm_Y = \delta_{g,e} \int f \, dm_Y = \mathrm{tr}'\left(\pi(f) w_g\right).$$

A trace-preserving $*$-isomorphism between weakly dense unital $*$-subalgebras of finite von Neumann algebras with faithful normal traces extends to a normal $*$-isomorphism of the completions: it extends to a unitary of the L^2-spaces intertwining left multiplication. The inverse is the gauge transformation associated with $(\overline{\phi_k})_k$. $\square$

3 The construction

3.1 The theta-characteristic cocycle

Lemma 3.1. *For every $h \in Q$ and $v \in \mathbb{Z}^4$:*

$$q_0(h^T v) \equiv q_0(v) + \langle W(h), v\rangle \pmod 2. \tag{6}$$

Proof. Write $v = (a, b)$ and $h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, so that $h^T v = (A^T a + C^T b,\ B^T a + D^T b)$ and

$$q_0(h^T v) = \langle A^T a + C^T b, B^T a + D^T b\rangle = a^T(AB^T)a + b^T(CD^T)b + a^T(AD^T)b + b^T(CB^T)a.$$

For a symmetric integer matrix S and an integer vector x one has

$$x^T S x = \sum_i S_{ii} x_i^2 + 2\sum_{i<j} S_{ij} x_i x_j \equiv \langle \operatorname{diag} S, x\rangle \pmod 2,$$

using $x_i^2 \equiv x_i$. By (1) the matrices AB^T and CD^T are symmetric, so the first two terms contribute $\langle \operatorname{diag}(AB^T), a\rangle + \langle \operatorname{diag}(CD^T), b\rangle = \langle W(h), v\rangle$ modulo 2. For the cross terms, note that $b^T(CB^T)a = a^T(BC^T)b$ (a transposed scalar), so by (1),

$$a^T(AD^T)b + b^T(CB^T)a = a^T(AD^T + BC^T)b = a^T(I + 2BC^T)b \equiv \langle a, b\rangle = q_0(v) \pmod 2. \ \square$$

Corollary 3.2. *$W(gh) \equiv W(g) + g\,W(h) \pmod{2\mathbb{Z}^4}$ for all $g, h \in Q$. Equivalently, $w := W \bmod 2$ is a 1-cocycle, $w \in Z^1(Q, \mathbb{F}_2^4)$, for the standard action of Q on $\mathbb{F}_2^4$.*

Proof. Since $(gh)^T = h^T g^T$, two applications of (6) give, for every $v \in \mathbb{Z}^4$,

$$q_0(v)+\langle W(gh), v\rangle \equiv q_0\big((gh)^T v\big) = q_0\big(h^T(g^T v)\big) \equiv q_0(g^T v)+\langle W(h), g^T v\rangle \equiv q_0(v)+\langle W(g)+gW(h), v\rangle$$

modulo 2. Since the pairing $\mathbb{F}_2^4 \times \mathbb{F}_2^4 \to \mathbb{F}_2$ is nondegenerate, the claim follows. Note also $W(e) = 0$, so w is normalized. □

Lemma 3.3. *The class $[w] \in H^1(Q, \mathbb{F}_2^4)$ is nonzero.*

Proof. For $u \in \mathbb{Z}^4$ let $T_u := I + u(Ju)^T$ be the symplectic transvection; it lies in Q and acts by $T_u x = x + \omega(x, u)u$, while $T_u^T v = v + \langle u, v\rangle Ju$. Put $c := \langle u, v\rangle$. Using (6), (2), $J^2 = -I$, $\omega(v, Ju) = v^T J^2 u = -\langle v, u\rangle$, and $c^2 \equiv c \pmod 2$,

$$\langle w(T_u), v\rangle \equiv q_0(T_u^T v)-q_0(v) \equiv c^2 q_0(Ju)+c\,\omega(v, Ju) \equiv c\,q_0(Ju)+c \equiv \langle u, v\rangle\big(q_0(Ju)+1\big) \pmod 2.$$

Suppose w were a coboundary, i.e. $w(g) \equiv (I + g)v_0 \pmod 2$ for some $v_0 \in \mathbb{Z}^4$ and all $g \in Q$. Then

$$\langle w(T_u), v\rangle \equiv \langle v_0, (I + T_u^T)v\rangle = \langle v_0, 2v + \langle u, v\rangle Ju\rangle \equiv \langle u, v\rangle\langle v_0, Ju\rangle \pmod 2.$$

For each $u \not\equiv 0 \pmod 2$ choose v with $\langle u, v\rangle$ odd; comparing the two displays gives

$$q_0(Ju) + 1 \equiv \langle v_0, Ju\rangle \pmod 2 \qquad \text{for all } u \not\equiv 0 \pmod 2.$$

Since $u \mapsto Ju$ is a bijection of $\mathbb{F}_2^4$, substituting $\tilde{u} := Ju$ yields

$$q_0(\tilde{u}) \equiv 1 + \langle v_0, \tilde{u}\rangle \pmod 2 \qquad \text{for all } \tilde{u} \in \mathbb{F}_2^4 \setminus \{0\}.$$

We count zeros on $\mathbb{F}_2^4 \setminus \{0\}$. For $q_0(\tilde{u}) = \tilde{u}_1\tilde{u}_3 + \tilde{u}_2\tilde{u}_4$: if $(\tilde{u}_1, \tilde{u}_2) = (0, 0)$ then $q_0 = 0$, giving 4 solutions; if $(\tilde{u}_1, \tilde{u}_2) \neq (0, 0)$ (three choices) then q_0 is a nonzero linear form in $(\tilde{u}_3, \tilde{u}_4)$ and vanishes for 2 of the 4 values, giving 6 solutions. Hence q_0 vanishes at 10 vectors, i.e. at exactly 9 nonzero vectors. On the other hand,

$$\{\tilde{u} \neq 0 :\ 1 + \langle v_0, \tilde{u}\rangle = 0\} = \{\tilde{u} \neq 0 :\ \langle v_0, \tilde{u}\rangle = 1\}$$

has 8 elements if $v_0 \not\equiv 0$ and 0 elements if $v_0 \equiv 0$. As $9 \notin \{0, 8\}$, no such v_0 exists. □

3.2 The torsion class $[\tau]$

By Corollary 3.2, $\partial W(g, h) := W(g) + gW(h) - W(gh) \in 2\mathbb{Z}^4$, so we may define

$$\tau(g, h) := \tfrac{1}{2}\big(W(g) + gW(h) - W(gh)\big) \in \mathbb{Z}^4 \qquad (g, h \in Q). \tag{7}$$

Since $2\tau = \partial W$ and $\partial\partial = 0$, the map τ is a 2-cocycle, $\tau \in Z^2(Q, \mathbb{Z}^4)$, normalized because $W(e) = 0$; moreover $2[\tau] = [\partial W] = 0$. By construction $[\tau] = \beta[w]$ is the Bockstein of $[w]$ for the coefficient sequence $0 \to \mathbb{Z}^4 \xrightarrow{2} \mathbb{Z}^4 \to \mathbb{F}_2^4 \to 0$.

Lemma 3.4. *$H^1(Q, \mathbb{Z}^4) = 0$, for the standard module $\mathbb{Z}^4$.*

Proof. (i) *Vanishing of invariants mod n.* Let $n \geq 1$ and suppose $v \in (\mathbb{Z}/n)^4$ is fixed by all transvections T_u ($u \in \mathbb{Z}^4$). Then $\omega(v,u)u \equiv 0 \pmod{n}$ for all u; taking $u = e_i$ gives $\omega(v, e_i) \equiv 0$ for $i = 1, \dots, 4$, i.e. $v^T J \equiv 0$, hence $v \equiv 0$ because J is invertible over $\mathbb{Z}$. Thus $H^0(Q, (\mathbb{Z}/n)^4) = 0$ for every n. The long exact sequence associated with $0 \to \mathbb{Z}^4 \xrightarrow{n} \mathbb{Z}^4 \to (\mathbb{Z}/n)^4 \to 0$ contains

$$0 = H^0(Q, (\mathbb{Z}/n)^4) \longrightarrow H^1(Q, \mathbb{Z}^4) \xrightarrow{\ n\ } H^1(Q, \mathbb{Z}^4),$$

so multiplication by n is injective on $H^1(Q, \mathbb{Z}^4)$ for every n: the group $H^1(Q, \mathbb{Z}^4)$ is torsion-free.

(ii) *Rank zero.* $Q = \mathrm{Sp}_4(\mathbb{Z})$ is a lattice in the simple Lie group $\mathrm{Sp}_4(\mathbb{R})$ of real rank 2, and $H^1(Q, \mathbb{R}^4) = 0$ for the restriction to Q of the finite-dimensional representation $\mathbb{R}^4$, by the vanishing theorem for the first cohomology of higher-rank lattices with finite-dimensional coefficients (Margulis [4, Ch. VII]; see also Borel–Wallach [2]). Since Q is finitely presented (being arithmetic), choose a finite presentation with generating set S and relator set R; for any Q-module M the groups $H^0(Q, M)$ and $H^1(Q, M)$ are computed by the finite Fox complex $M \to M^S \to M^R$, functorially in M and compatibly with flat base change. In particular $H^1(Q, \mathbb{Z}^4)$ is a finitely generated abelian group and

$$H^1(Q, \mathbb{Z}^4) \otimes_{\mathbb{Z}} \mathbb{R} \ \cong\ H^1(Q, \mathbb{R}^4) = 0.$$

A finitely generated torsion-free abelian group of rank 0 is trivial. $\square$

Proposition 3.5. *$[\tau] \neq 0$ in $H^2(Q, \mathbb{Z}^4)$, and $2[\tau] = 0$.*

Proof. The Bockstein sequence gives exactness of

$$H^1(Q, \mathbb{Z}^4) \longrightarrow H^1(Q, \mathbb{F}_2^4) \xrightarrow{\ \beta\ } H^2(Q, \mathbb{Z}^4).$$

By Lemma 3.4 the left-hand group vanishes, so β is injective; by Lemma 3.3, $[\tau] = \beta[w] \neq 0$. The relation $2[\tau] = 0$ was noted above. $\square$

3.3 The two groups

Inside $\widehat{\Gamma} := \frac{1}{2}\mathbb{Z}^4 \rtimes Q$ define

$$\Gamma_0 := \mathbb{Z}^4 \rtimes Q = \{(n, g) : n \in \mathbb{Z}^4,\ g \in Q\}, \qquad \Gamma_\tau := \Big\{\big(n + \tfrac{1}{2}W(g),\, g\big) : n \in \mathbb{Z}^4,\ g \in Q\Big\}.$$

Lemma 3.6. *(i) Γ_τ is a subgroup of $\widehat{\Gamma}$, and $1 \to \mathbb{Z}^4 \to \Gamma_\tau \to Q \to 1$ (kernel $\{(n, I) : n \in \mathbb{Z}^4\}$, projection $(v, g) \mapsto g$) is an extension of Q by the standard module $\mathbb{Z}^4$, with class $[\tau]$.*

(ii) Γ_0 and Γ_τ both have index 16 in $\widehat{\Gamma}$, and

$$\Gamma_0 \cap \Gamma_\tau = \mathbb{Z}^4 \rtimes \Gamma_{1,2}, \qquad \Gamma_{1,2} := \{g \in Q : W(g) \equiv 0 \bmod 2\},$$

where $\Gamma_{1,2}$ (Igusa's theta group) has finite index in Q.

Proof. (i) By (7),

$$\big(n+\tfrac{1}{2}W(g), g\big)\big(n'+\tfrac{1}{2}W(h), h\big) = \Big(n+gn'+\tfrac{1}{2}\big(W(g)+gW(h)\big),\, gh\Big) = \Big(n+gn'+\tau(g,h)+\tfrac{1}{2}W(gh),\, gh\Big),$$

which lies in Γ_τ; a similar computation handles inverses (using $W(g^{-1}) \equiv -g^{-1}W(g)$ modulo $2\mathbb{Z}^4$, which follows from Corollary 3.2). The map $s(g) := (\frac{1}{2}W(g), g)$ is a section with

$s(e)=e$, and the above computation gives $s(g)s(h)s(gh)^{-1}=(\tau(g,h),I)$, so the extension class is $[\tau]$. Conjugation by $s(g)$ maps (n,I) to (gn,I), so the module structure on the kernel is the standard one.

(ii) Both Γ_0 and Γ_τ surject onto Q with kernel $\mathbb{Z}^4$, so both have index $[\frac{1}{2}\mathbb{Z}^4:\mathbb{Z}^4]=16$ in $\widehat{\Gamma}$ (cosets represented by the translations (t,I), $t\in\frac{1}{2}\mathbb{Z}^4/\mathbb{Z}^4$). An element (n,g) with $n\in\mathbb{Z}^4$ lies in Γ_τ if and only if $\frac{1}{2}W(g)\in\mathbb{Z}^4$, i.e. $w(g)=0$; this proves the description of $\Gamma_0\cap\Gamma_\tau$. By Corollary 3.2, $\Gamma_{1,2}$ is a subgroup ($w(gh)=w(g)+gw(h)$ and $w(g^{-1})=-g^{-1}w(g)$), and $w(g)$ depends only on g modulo 2 (the entries of AB^T and CD^T modulo 2 depend only on A,B,C,D modulo 2), so $\Gamma_{1,2}$ contains the principal congruence subgroup of level 2 and therefore has finite index in Q. □

Lemma 3.7. Γ_0 *and* Γ_τ *are ICC.*

Proof. Write Γ_c for $c\in\{0,\tau\}$; all computations take place inside $\widehat{\Gamma}$. Let $(v,g)\in\Gamma_c$, $(v,g)\neq e$.

Case $g\neq\pm I$. The image in Q of the Γ_c-conjugacy class of (v,g) is the full Q-conjugacy class of g, so it suffices to show $[Q:C_Q(g)]=\infty$. If the centralizer had finite index, then g would centralize a finite-index subgroup $Q_1\leq Q$. A finite-index subgroup of a lattice is again a lattice, hence Zariski dense in $\mathrm{Sp}_4(\mathbb{R})$ by the Borel density theorem; thus g would be central in $\mathrm{Sp}_4(\mathbb{R})$, i.e. $g=\pm I$, a contradiction.

Case $g=I$, $v\neq 0$. Conjugating (v,I) by an element of Γ_c lying over $h\in Q$ yields (hv,I). If the orbit Qv were finite, the stabilizer of v would have finite index in Q, hence be Zariski dense, so $\mathrm{Sp}_4(\mathbb{R})$ would fix $v\neq 0$; but the standard representation has no nonzero invariant vector.

Case $g=-I$. Note $W(-I)=0$, so the elements of Γ_c over $-I$ are exactly the $(v,-I)$ with $v\in\mathbb{Z}^4$. For $m\in\mathbb{Z}^4$,

$$(m,I)(v,-I)(m,I)^{-1}=(v+2m,\,-I),$$

giving infinitely many conjugates. □

Lemma 3.8. Γ_0 *and* Γ_τ *have Kazhdan's property (T).*

Proof. The group $\mathrm{Sp}_4(\mathbb{Z})$ has property (T) (Kazhdan), and the pair $(\mathbb{Z}^4\rtimes\mathrm{Sp}_4(\mathbb{Z}),\mathbb{Z}^4)$ has relative property (T) (Burger's criterion); consequently $\Gamma_0=\mathbb{Z}^4\rtimes\mathrm{Sp}_4(\mathbb{Z})$ has property (T). This is a standard example, see Bekka–de la Harpe–Valette [1], where $\mathbb{Z}^{2g}\rtimes\mathrm{Sp}_{2g}(\mathbb{Z})$, $g\geq 2$, appears as a Kazhdan group. (Alternatively, $\mathbb{Z}^4\rtimes\mathrm{Sp}_4(\mathbb{Z})$ is a lattice in $\mathbb{R}^4\rtimes\mathrm{Sp}_4(\mathbb{R})$, which has (T) because $\mathrm{Sp}_4(\mathbb{R})$ has (T) and $(\mathbb{R}^4\rtimes\mathrm{Sp}_4(\mathbb{R}),\mathbb{R}^4)$ has relative (T).)

Property (T) passes to finite-index subgroups and to finite-index overgroups. By Lemma 3.6(ii), $\mathbb{Z}^4\rtimes\Gamma_{1,2}$ has finite index in Γ_0, hence has (T); and it has finite index in Γ_τ, hence Γ_τ has (T). □

4 Non-isomorphism

Lemma 4.1 (Recognition of the fiber)**.** *Let $c\in\{0,\tau\}$ and let $A_c\leq\Gamma_c$ denote the preimage of $\{\pm I\}\leq Q$. Then:*

(i) A_c is the amenable radical of Γ_c; in particular it is characteristic.

(ii) The subgroup generated by the squares of elements of A_c equals $2\mathbb{Z}^4$, and

$$\mathbb{Z}^4=C_{A_c}\big(\langle a^2:a\in A_c\rangle\big).$$

Consequently, every isomorphism $\Phi:\Gamma_0\to\Gamma_\tau$ satisfies $\Phi(\mathbb{Z}^4)=\mathbb{Z}^4$.

Proof. (i) Since $\{\pm I\}$ is the center of Q, it is normal, hence A_c is normal in Γ_c; and A_c is amenable, being an extension of $\mathbb{Z}/2$ by $\mathbb{Z}^4$. Conversely let $R \lhd \Gamma_c$ be amenable and let $\bar{R} \lhd Q$ be its image, a normal amenable subgroup of Q. Being amenable and linear, $\bar{R}$ is virtually solvable by the Tits alternative, hence its Zariski closure $H \leq \mathrm{Sp}_4(\mathbb{R})$ is virtually solvable (the Zariski closure of a solvable group is solvable). Since Q is Zariski dense in $\mathrm{Sp}_4(\mathbb{R})$ (Borel density) and $\bar{R}$ is normalized by Q, the subgroup H is normalized by the Zariski closure of Q, i.e. $H \lhd \mathrm{Sp}_4(\mathbb{R})$. The Lie algebra of H is an ideal of the simple Lie algebra $\mathfrak{sp}_4(\mathbb{R})$, hence is 0 or everything; the latter is impossible since $\mathrm{Sp}_4(\mathbb{R})$ is not virtually solvable. So H is finite and normal in the connected group $\mathrm{Sp}_4(\mathbb{R})$, hence central: $H \subseteq \{\pm I\}$. Therefore $\bar{R} \subseteq \{\pm I\}$ and $R \subseteq A_c$. Thus $\mathrm{Rad}_{\mathrm{am}}(\Gamma_c) = A_c$.

(ii) As noted in the proof of Lemma 3.7, $W(-I) = 0$, so in both groups $A_c = \{(m, I) : m \in \mathbb{Z}^4\} \cup \{(v, -I) : v \in \mathbb{Z}^4\}$. Squares: $(m, I)^2 = (2m, I)$ and $(v, -I)^2 = (v + (-I)v, I) = (0, I)$; hence $\langle a^2 : a \in A_c \rangle = 2\mathbb{Z}^4$. Centralizer: $(v, -I)(2n, I)(v, -I)^{-1} = (-2n, I) \neq (2n, I)$ for $n \neq 0$, whereas every (m, I) commutes with $2\mathbb{Z}^4$. Hence $C_{A_c}(2\mathbb{Z}^4) = \{(m, I) : m \in \mathbb{Z}^4\} = \mathbb{Z}^4$.

Finally, an isomorphism $\Phi : \Gamma_0 \to \Gamma_\tau$ maps amenable radical to amenable radical, i.e. $\Phi(A_0) = A_\tau$; hence it maps the subgroup generated by squares of A_0 onto that of A_τ, i.e. $\Phi(2\mathbb{Z}^4) = 2\mathbb{Z}^4$; hence it maps the corresponding centralizers onto each other: $\Phi(\mathbb{Z}^4) = \mathbb{Z}^4$. $\square$

Theorem 4.2. $\Gamma_0 \not\cong \Gamma_\tau$.

Proof. Suppose $\Phi : \Gamma_0 \to \Gamma_\tau$ is an isomorphism. By Lemma 4.1, $\Phi(\mathbb{Z}^4) = \mathbb{Z}^4$, so Φ descends to an automorphism α of the quotient Q; that is, $p_\tau \circ \Phi = \alpha \circ p_0$, where p_0, p_τ are the projections onto Q.

Let $s : Q \to \Gamma_0$, $s(g) = (0, g)$, be the splitting homomorphism of $\Gamma_0 \to Q$, and put $s' := \Phi \circ s \circ \alpha^{-1}$. Then $p_\tau \circ s' = \alpha \circ p_0 \circ s \circ \alpha^{-1} = \mathrm{id}_Q$, so s' is a section of $\Gamma_\tau \to Q$; moreover it is a homomorphism, being a composition of homomorphisms. Hence the extension

$$1 \to \mathbb{Z}^4 \to \Gamma_\tau \to Q \to 1$$

splits. Since the conjugation action of Γ_τ on its kernel $\mathbb{Z}^4$ is the standard Q-action (Lemma 3.6(i)), splitting means exactly that its class $[\tau] \in H^2(Q, \mathbb{Z}^4)$ vanishes. This contradicts Proposition 3.5. $\square$

5 Isomorphism of the group von Neumann algebras

5.1 The common crossed-product picture

We apply Section 2.3 to both extensions, with kernel $N = \mathbb{Z}^4$ and quotient $B = Q$. Here

$$Y = \widehat{\mathbb{Z}^4} = \mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4, \qquad u_n(x) = e^{2\pi i \langle n, x\rangle},$$

and the dual action of Q is

$$g \cdot x = (g^T)^{-1} x, \qquad \text{equivalently} \qquad g^{-1} \cdot x = g^T x \pmod{\mathbb{Z}^4},$$

which preserves Haar measure since $|\det g| = 1$. We write σ for this action and $T_g f = f(g^{-1}\cdot)$; thus $T_g u_n = u_{gn}$.

For Γ_0 take the splitting section $g \mapsto (0, g)$; its cocycle is trivial, so the associated $\mathcal{U}(L^\infty(\mathbb{T}^4))$-valued cocycle is $\mu_0 \equiv 1$. For Γ_τ take $s(g) = (\frac{1}{2}W(g), g)$; by Lemma 3.6(i) its cocycle is τ, so the associated cocycle is $\mu_\tau(g, h) = u_{\tau(g,h)}$. By Lemma 2.3,

$$L(\Gamma_0) \cong L^\infty(\mathbb{T}^4) \rtimes_\sigma Q \ \ (\text{untwisted}), \qquad L(\Gamma_\tau) \cong L^\infty(\mathbb{T}^4) \rtimes_{\sigma, u_\tau} Q. \tag{8}$$

5.2 The carry cocycle

Let $\ell : \mathbb{T}^4 \to [0,1)^4$ be the canonical Borel branch of the quotient map $\mathbb{R}^4 \to \mathbb{T}^4$ (i.e. $\ell(x)$ is the unique representative of x in $[0,1)^4$). Define the *carry cocycle*

$$\kappa(g,x) := \ell(g^{-1}x) - g^T\ell(x) \in \mathbb{Z}^4 \qquad (g \in Q,\ x \in \mathbb{T}^4).$$

It is an integer vector because $g^{-1}x = g^T x$ modulo $\mathbb{Z}^4$; for fixed g, the map $x \mapsto \kappa(g,x)$ is Borel and bounded. Applying the definition to $(gh)^{-1}x$ and using $(gh)^T = h^T g^T$ together with $\ell(g^{-1}x) = g^T\ell(x) + \kappa(g,x)$, one finds the cocycle identity

$$\kappa(gh,x) = h^T\kappa(g,x) + \kappa(h, g^{-1}x). \tag{9}$$

Note that $\kappa(e,\cdot) = 0$.

5.3 The explicit gauge

Theorem 5.1 (Untwisting)**.** *Define Borel functions $\Psi_g : \mathbb{T}^4 \to \mathbb{T} \subset \mathbb{C}$, $g \in Q$, by*

$$\Psi_g(x) := \exp\Big(\pi i \Big[\ \langle W(g), \ell(x)\rangle\ +\ q_0\big(\kappa(g,x)\big)\ +\ \omega\big(\kappa(g,x),\ g^T\ell(x)\big)\ \Big]\Big).$$

Then $\Psi_e = 1$, each $\Psi_g \in \mathcal{U}(L^\infty(\mathbb{T}^4))$, and for all $g, h \in Q$ and all $x \in \mathbb{T}^4$,

$$\Psi_g(x)\,\Psi_h(g^{-1}x)\,\overline{\Psi_{gh}(x)}\ =\ u_{\tau(g,h)}(x)\ =\ e^{2\pi i\langle \tau(g,h), x\rangle}.$$

Proof. Denote by $P_g(x)$ the real number in square brackets, so $\Psi_g = e^{\pi i P_g}$, and abbreviate

$$\ell := \ell(x), \quad y := g^{-1}x, \quad \kappa_g := \kappa(g,x), \quad \kappa_h := \kappa(h,y), \quad \kappa_{gh} := \kappa(gh,x).$$

By the definition of κ and by (9),

$$\ell(y) = g^T\ell + \kappa_g, \qquad \kappa_{gh} = h^T\kappa_g + \kappa_h. \tag{10}$$

We must show that

$$P_g(x) + P_h(y) - P_{gh}(x) \in 2\langle \tau(g,h), \ell\rangle + 2\mathbb{Z}.$$

We treat the three groups of terms separately.

(A) Linear terms. Using (10), $\langle W(h), g^T\ell\rangle = \langle gW(h), \ell\rangle$, and $W(g) + gW(h) - W(gh) = 2\tau(g,h)$:

$$\langle W(g), \ell\rangle + \langle W(h), \ell(y)\rangle - \langle W(gh), \ell\rangle = 2\langle \tau(g,h), \ell\rangle + \langle W(h), \kappa_g\rangle.$$

The remainder $\langle W(h), \kappa_g\rangle$ is an *integer.*

(B) Quadratic (Arf) terms. All quantities here are integers; we compute modulo 2. By (10), (2), and Lemma 3.1 applied to h and $v = \kappa_g$,

$$q_0(\kappa_{gh}) = q_0\big(h^T\kappa_g + \kappa_h\big) \equiv q_0(h^T\kappa_g) + q_0(\kappa_h) + \omega(h^T\kappa_g, \kappa_h) \equiv q_0(\kappa_g) + \langle W(h), \kappa_g\rangle + q_0(\kappa_h) + \omega(h^T\kappa_g, \kappa_h)$$

modulo 2. Hence

$$q_0(\kappa_g) + q_0(\kappa_h) - q_0(\kappa_{gh}) \equiv \langle W(h), \kappa_g\rangle + \omega(h^T\kappa_g, \kappa_h) \pmod 2. \tag{11}$$

(C) Cross terms. These cancel *exactly in* $\mathbb{R}$ up to a single integral term. Using (10), $(gh)^T = h^T g^T$, and the Sp-invariance $\omega(h^T u, h^T v) = u^T hJh^T v = \omega(u,v)$ (valid because $hJh^T = J$):

$$\begin{aligned}
\omega\big(\kappa_h,\ h^T\ell(y)\big) &= \omega\big(\kappa_h,\ h^T g^T\ell\big) + \omega\big(\kappa_h,\ h^T\kappa_g\big),\\
\omega\big(\kappa_{gh},\ (gh)^T\ell\big) &= \omega\big(h^T\kappa_g,\ h^T g^T\ell\big) + \omega\big(\kappa_h,\ h^T g^T\ell\big) = \omega\big(\kappa_g,\ g^T\ell\big) + \omega\big(\kappa_h,\ h^T g^T\ell\big).
\end{aligned}$$

Therefore

$$\omega\big(\kappa_g, g^T\ell\big) + \omega\big(\kappa_h, h^T\ell(y)\big) - \omega\big(\kappa_{gh}, (gh)^T\ell\big) = \omega\big(\kappa_h, h^T\kappa_g\big) = -\,\omega\big(h^T\kappa_g, \kappa_h\big),$$

an *integer*.

Total. Adding (A), (B), (C):

$$P_g(x) + P_h(y) - P_{gh}(x) = 2\langle \tau(g,h), \ell\rangle + E,$$

where

$$E = \langle W(h), \kappa_g\rangle + \big[q_0(\kappa_g) + q_0(\kappa_h) - q_0(\kappa_{gh})\big] - \omega\big(h^T\kappa_g, \kappa_h\big) \in \mathbb{Z}.$$

By (11) and $-1 \equiv 1 \pmod 2$,

$$E \equiv \langle W(h), \kappa_g\rangle + \langle W(h), \kappa_g\rangle + \omega(h^T\kappa_g, \kappa_h) + \omega(h^T\kappa_g, \kappa_h) \equiv 0 \pmod 2.$$

Hence $P_g(x) + P_h(g^{-1}x) - P_{gh}(x) \in 2\langle \tau(g,h), \ell(x)\rangle + 2\mathbb{Z}$ and

$$\Psi_g(x)\Psi_h(g^{-1}x)\overline{\Psi_{gh}(x)} = e^{\pi i\left(P_g(x)+P_h(g^{-1}x)-P_{gh}(x)\right)} = e^{2\pi i\langle \tau(g,h),\ell(x)\rangle} = u_{\tau(g,h)}(x),$$

the last equality because $\tau(g,h) \in \mathbb{Z}^4$, so that $e^{2\pi i\langle \tau(g,h),\cdot\rangle}$ does not depend on the choice of branch. Finally $\Psi_e = 1$ because $W(e) = 0$ and $\kappa(e,\cdot) = 0$. □

Theorem 5.2. *$L(\Gamma_0) \cong L(\Gamma_\tau)$, by a trace-preserving isomorphism which restricts to the identity on $L^\infty(\mathbb{T}^4)$ in the pictures* (8).

Proof. Since $(T_g\Psi_h)(x) = \Psi_h(g^{-1}x)$, Theorem 5.1 says precisely that

$$\mu_\tau(g,h) = u_{\tau(g,h)} = \Psi_g\, T_g(\Psi_h)\, \mu_0(g,h)\, \overline{\Psi_{gh}} \qquad (g,h \in Q),$$

with $\mu_0 \equiv 1$; that is, condition (5) holds with $\phi_g = \Psi_g$, and $\Psi_e = 1$. By Lemma 2.4,

$$L^\infty(\mathbb{T}^4) \rtimes_\sigma Q \;\cong\; L^\infty(\mathbb{T}^4) \rtimes_{\sigma,u_\tau} Q,$$

trace-preservingly and identically on $L^\infty(\mathbb{T}^4)$. Combining with (8) finishes the proof. □

6 Proof of the Main Theorem

Proof of Theorem 1.1. Take

$$\Gamma := \Gamma_0 = \mathbb{Z}^4 \rtimes \mathrm{Sp}_4(\mathbb{Z}), \qquad \Lambda := \Gamma_\tau.$$

Both are countable. They are ICC by Lemma 3.7, so $L(\Gamma)$ and $L(\Lambda)$ are II_1 factors by Lemma 2.1. They have property (T) by Lemma 3.8. They are non-isomorphic by Theorem 4.2. Finally, $L(\Gamma) \cong L(\Lambda)$ by Theorem 5.2. □

In particular, Connes' rigidity conjecture for ICC property (T) groups is false.

7 Remarks

Remark 7.1 (What makes the construction work)**.** The class $[\tau]$ is a nonzero 2-*torsion* class. Torsion extension classes with values in a module whose dual group is *connected* (here $\widehat{\mathbb{Z}^4} = \mathbb{T}^4$) are invisible to the group von Neumann algebra, because the "half-characters" $e^{\pi i \langle W(g), \cdot \rangle}$, which do not exist as elements of the group, do exist as measurable functions on the dual — at the cost of the branch defects $\kappa(g,x)$. The content of Theorem 5.1 is that these branch defects can be repaired *exactly* by the two correction terms $q_0(\kappa)$ and $\omega(\kappa, g^T\ell)$: the first converts the theta-cocycle pairing $\langle W(h), \kappa_g \rangle$ into the symplectic "carry" pairing $\omega(h^T\kappa_g, \kappa_h)$, via the transformation law (6) of the quadratic refinement q_0 — which is the invariant meaning of the theta-characteristic cocycle — while the second cancels that pairing identically in $\mathbb{R}$.

The simplest illustration of the phenomenon is $L(\mathbb{Z}\times\mathbb{Z}/k) \ncong L(\mathbb{Z})$ versus $L(\mathbb{Z}) \cong L^\infty(\mathbb{T})$: if $\mathbb{Z}$ is viewed as the non-split extension of $\mathbb{Z}/k$ by $\mathbb{Z}$, then the corresponding twisted crossed product over $\mathbb{T}$ is untwisted by the measurable k-th root $x \mapsto e^{2\pi i \ell(x)/k}$ of the canonical unitary; a von Neumann algebra cannot tell whether a k-th root of a canonical unitary is a group element or merely a measurable function. Theorem 5.1 is the property (T)-compatible incarnation of this: the groups Γ_0 and Γ_τ are even commensurable (Lemma 3.6(ii)); they differ only in which square roots of the unitaries $u_{W(g)}$ are group elements.

Remark 7.2 (Variants)**.** For any $k \geq 1$ the same proof gives

$$L\big(\mathbb{Z}^{4k} \rtimes Q\big) \cong L\big(\Gamma_{\tau^{\oplus k}}\big)$$

for the diagonal action, with gauge $x = (x_1, \dots, x_k) \mapsto \prod_{j=1}^k \Psi_g(x_j)$; all the side conditions (ICC, property (T)) persist. For $k \geq 2$ one can moreover distinguish pairs of *non-split* extensions, e.g. $\Gamma_{\tau\oplus 0} \ncong \Gamma_{\tau\oplus\tau}$: since $\mathrm{End}_Q(\mathbb{Z}^4) = \mathbb{Z}$ (absolute irreducibility of the standard representation), an isomorphism compatible with the fibers acts on $H^2(Q, \mathbb{Z}^4)^{\oplus k}$ through $\mathrm{GL}_k(\mathbb{Z})$, and a matrix carrying $(\tau, 0, \dots)$ to $(\tau, \tau, 0 \dots)$ coordinatewise modulo 2 would need all row sums even, hence even determinant — impossible. Finally, Sp_4 may be replaced by Sp_{2g} for any $g \geq 2$, with $W(g) = (\mathrm{diag}(AB^T), \mathrm{diag}(CD^T)) \in \mathbb{Z}^{2g}$ verbatim.

Remark 7.3 (Relation to rigidity results)**.** The known W*-superrigidity theorems concern classes of groups of wreath-like or amalgam type (Ioana–Popa–Vaes, Berbec–Vaes, Chifan–Ioana–Osin–Sun, Donvil–Vaes) and do not cover $\mathbb{Z}^4 \rtimes \mathrm{Sp}_4(\mathbb{Z})$; the example above shows that no such theorem can hold for this group. The gauge Ψ is a genuinely non-linear measurable phase: its essential ingredients are the quadratic term $q_0(\kappa)$ and the symplectic carry term $\omega(\kappa, g^T\ell)$, and the construction makes essential use of the finite-rank, non-permutation module $\mathbb{Z}^4$.

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