Task 1.1: Detailed Review of the Thrust/HJM Shoulder Paper

Paper: "Sudakov Shoulder Resummation for Thrust and Heavy Jet Mass"
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1. Executive Summary

This paper provides the first complete NLL resummation of Sudakov shoulder logarithms for thrust (τ) and heavy jet mass (ρ) in e^+e^- annihilation. The key insights relevant for extending to the C-parameter are:

1. **Origin of shoulder logs:** Logarithms arise from kinematic configurations with narrow jets near the symmetric trijet configuration
2. **Factorization structure:** A SCET-based factorization formula involving a trijet hard function, three jet functions, and a 6-sextant soft function
3. **No non-global logarithms:** Despite the global nature of the observables, NGLs are absent due to continuity arguments
4. **Sudakov–Landau pole:** A singularity appears in the resummed distribution at $\eta_\ell + \eta_h = 1$, which is canceled by power corrections

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2. Observable Definitions and Key Kinematics

2.1 Thrust Definition
```
T = max_n̂ [ Σ_j |p⃗_j · n̂| / Σ_j |p⃗_j| ]
τ = 1 − T
```

For 3 massless partons with invariants $s_{ij} = (p_i + p_j)^2/Q^2$:
```
τ = min(s₁₂, s₁₃, s₂₃) ≤ 1/3
```

2.2 Heavy Jet Mass Definition
```
ρ = (1/Q²) max(m₁², m₂²)
```

where m_1, m_2 are the hemisphere invariant masses.

2.3 Shoulder Variables
- **Left shoulder (HJM):** $r \equiv 1/3 - \rho > 0$
- **Right shoulder (thrust/HJM):** $t = \tau - 1/3 > 0$ or $s = \rho - 1/3 > 0$

2.4 Symmetric Trijet Configuration
At $s_{12} = s_{13} = s_{23} = 1/3$:
- $\tau = \rho = 1/3$
- Three partons at 120° separation with equal energies $E = Q/3$
- Momenta: $p_1 = (Q/3)(1, 0, 0, 1)$, $p_2 = (Q/3)(1, 0, \sqrt{3}/2, -1/2)$, $p_3 = (Q/3)(1, 0, -\sqrt{3}/2, -1/2)$

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3. Key Distinction: Kink vs. Step Discontinuity

Critical insight for C-parameter:

Observable	Behavior at LO Boundary	Type

Thrust (τ)	Discontinuity in $d\sigma/d\tau$	Kink
Heavy jet mass (ρ)	Discontinuity in $d\sigma/d\rho$	Kink
C-parameter	**Discontinuity in σ itself**	**Step**

For thrust/HJM at LO:
```
(1/σ₀)(dσ/dτ) ∝ (1/3 − τ) θ(1/3 − τ)
```
The distribution is linear in $(1/3 - \tau)$, giving a kink (discontinuity in first derivative).

C-parameter (to be verified in Tasks 1.7–1.9) has:
```
(1/σ₀)(dσ/dC) ∝ f(C) θ(3/4 − C)
```
where $f(3/4) \neq 0$, giving a step discontinuity.

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4. NLO Fixed-Order Analysis (Section II of BSZ)

4.1 Four-Parton Kinematics

Phase space variables used:
```
s₂₃₄ = (p₂ + p₃ + p₄)²/Q²      (hard variable)
s₃₄ = (p₃ + p₄)²/Q²           (jet mass)
z = collinear fraction
ω = (1/2) n̄·(p₃ + p₄)/Q
φ = azimuthal angle
```

Power counting for shoulder region ($r \sim \lambda \ll 1$):
– Collinear: $s_{34} \sim \lambda$, $z \sim \lambda^0$
– Soft: $z \sim \lambda$
– Soft-collinear: $z \sim \lambda$, $s_{34} \sim \lambda$

4.2 Key Finding: Which Regions Contribute Logs

For left shoulder of HJM ($r = 1/3 - \rho > 0$):
– Only T_{12} maximal regions (2 partons in each hemisphere) contribute logs
– T_1 maximal regions (3 partons in heavy hemisphere) do NOT contribute logs

For right shoulder of thrust/HJM:
– Only T_1 maximal regions (1 parton in light hemisphere) contribute

4.3 Matrix Element Structure

Collinear limit ($p_3 \parallel p_4$):
```
|M_coll|² ~ |M₀|² (g⁴_s/s₃₄) × [splitting function]
```

Splitting functions depend on polarization of parent gluon:
– $\gamma* \to q\bar{q}g\bar{g}$: Includes $\cos(2\varphi)$ azimuthal dependence
– $\gamma* \to q\bar{q}q'\bar{q}'$: Also has azimuthal dependence

Soft limit (p_4 soft):
```
|M_soft|² ~ |M₀|² g⁴_s × [Eikonal factors with color structure]
```

4.4 NLO Logarithmic Coefficients (Eq. 46–49 in BSZ)

For left shoulder of HJM:
```

```
(1/σ₀)(dσ^(C_F²)/dr) = (α_s/4π)² C_F² r [−192 ln²r + (96 + 768 ln2 − 384 ln3) ln r + ...]

(1/σ₀)(dσ^(C_A)/dr) = (α_s/4π)² C_F C_A r [−96 ln²r + (16 + 384 ln2 − 192 ln3) ln r + ...]

(1/σ₀)(dσ^(n_f)/dr) = (α_s/4π)² C_F T_f n_f r [64 ln r + ...]
```

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5. Factorization Theorem (Section III of BSZ)

5.1 Recoil Sensitivity Issue

Simple convolution approach fails at NLL:
```
σ_resummed(ρ) ~ ∫ dm² σ_LO(ρ − m²) J(m²)   ← PROBLEMATIC
```

The shift ρ → ρ + m² vs ρ → ρ − 2m² depends on whether energy or momentum is held fixed.
This recoil ambiguity prevents simple convolution-based resummation.

5.2 Phase Space Factorization Approach

Key observation: Only configurations differing from trijet by soft/collinear emissions
contribute logs.

Measurement constraint for left shoulder HJM:
```
W(r, m_j, k_i) = r − m₁² + m₂² + m₃² + (soft contributions) > 0
```

where the soft contributions involve projections of soft momenta onto 6 sextant directions.

5.3 Six-Sextant Decomposition

Soft radiation divided into 6 regions (like orange segments):
− k₁, k₂, k₃: Sextants containing jets (projections p_j · k)
− k̄₁, k̄₂, k̄₃: Sextants between jets (projections v̄_j · k)

The projection vectors:
```
p_j = (Q/3) n_j        (lightlike)
v̄₁ = (Q/3) ñ₁          (lightlike, opposite to jet 1)
v̄₂, v̄₃                (spacelike, between jets)
```

5.4 Factorization Formula

```
(1/σ₁)(dσ/dr) = H(Q) ∫ d³m² d⁶q J(m₁²) J(m₂²) J(m₃²) S₆(q_i) W(m_j, q_i, r) θ[W]
```

Simplified with trijet hemisphere soft function S(q_ℓ, q_h):
```
(dσ/dr) = ∫ dm_h² dm_ℓ² (d²σ/dm_ℓ² dm_h²) (r + m_h² − m_ℓ²) θ(r + m_h² − m_ℓ²)
```

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6. One-Loop Ingredients (Section III.C-D of BSZ)

6.1 Soft Function Integrals

Four independent integrals I₁, I₂, I₃, I₄ distinguished by measurement region position
relative to Wilson lines:

```
I₁(q) = (1/q^{1+2ε}) [1/ε − (7/2)ln2 + ln3 − 3κ/(2π)]
I₂(q) = (1/q^{1+2ε}) [−ln2 + 3κ/π]
I₃(q) = (1/q^{1+2ε}) [ln2 + 3κ/π]
I₄(q) = (1/q^{1+2ε}) [(3/2)ln2 − 3κ/(2π)]
```

where κ = Im Li₂(e^{iπ/3}) ≈ 1.0149 (Gieseking's constant).

6.2 Soft Function Anomalous Dimensions

Gluon channel (light hemisphere = gluon jet):
```
γ^s_{qq} = −4C_F ln6
γ^s_g = −2C_A ln3 + 4C_F ln2
```

Quark channel (light hemisphere = quark jet):
```
γ^s_{qg} = −2(C_A + C_F) ln6
γ^s_q = −2C_F ln(3/2) + 2C_A ln2
```

6.3 Hard Function

The trijet hard function is related to direct photon/W/Z production:
```
H(Q, μ_h) = 1 + (α_s/4π)[−(2C_F + C_A)(Γ₀/4) ln²(Q²/μ_h²) − γ_h ln(Q²/μ_h²)]
```

with γ_h = −2(2C_F + C_A) ln3 − 6C_F − β₀

6.4 Jet Functions

Standard inclusive jet functions:
```
J_i(m², μ) ∝ (m²/μ_j²)^{η_j} with η_j = 2C_i A_Γ(μ_j, μ_s)
```

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7. Resummed Expressions (Section III.D of BSZ)

7.1 Master Formula Structure

Thrust (right shoulder, t = τ − 1/3 > 0):
```
(1/σ₁)(dσ/dt) = Π(∂_{η_ℓ}, ∂_{η_h}) t (tQ/μ_s)^{η_ℓ} (tQ/μ_s)^{η_h}
                × e^{−γ_E(η_ℓ+η_h)} / Γ(2 + η_ℓ + η_h)
```

Heavy jet mass left shoulder (r = 1/3 − ρ > 0):
```
(1/σ₁)(dσ/dr) = Π(...) r (rQ/μ_s)^{η_ℓ+η_h} × [sin(πη_ℓ)/sin(π(η_ℓ+η_h))] / Γ(2+η_ℓ+η_h)
```

Heavy jet mass right shoulder (s = ρ − 1/3 > 0):
```
(1/σ₁)(dσ/ds) = Π(...) s (sQ/μ_s)^{η_ℓ+η_h} × [sin(πη_h)/sin(π(η_ℓ+η_h))] / Γ(2+η_ℓ+η_h)
```

7.2 Anomalous Dimension Parameters

Gluon channel:
```

```
η_ℓ = 2C_A A_Γ(μ_j, μ_s)
η_h = 4C_F A_Γ(μ_j, μ_s)
```

Quark channel:
```
η_ℓ = 2C_F A_Γ(μ_j, μ_s)
η_h = 2(C_F + C_A) A_Γ(μ_j, μ_s)
```

7.3 Canonical Scale Choices
```
μ_h = Q,   μ_j = √r Q,   μ_s = r Q
```

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8. Important Physics Results

8.1 No Non-Global Logarithms

Key argument: Configurations with large jet masses can contribute to both r > 0 and r < 0, so they must be smooth across r = 0. Only soft/collinear configurations (small masses) can generate the non-analytic behavior.

Mathematically, separating UV and IR contributions:
```
∫_0^∞ dx ∫_0^∞ dy x^{a-1} y^{b-1} (r+y-x)θ(r+y-x) = [global part] + [regular part]
```

8.2 Sudakov-Landau Pole

The factor $\sin^{-1}(\pi(\eta_\ell + \eta_h))$ produces singularities when $\eta_\ell + \eta_h \in \mathbb{Z}$.

At LL with canonical scales:
```
η_ℓ + η_h = (α_s/4π)(C_A + 2C_F)Γ_0 ln r
```

The pole at η_ℓ + η_h = 1 occurs at:
```
r_pole = exp[-4π/((C_A + 2C_F)α_s Γ_0)] ≈ exp[-3π/(17α_s)]
```

For α_s = 0.119: r_pole ≈ 0.01 (LL) or ≈ 0.06 (NLL)

Resolution: Power corrections become O(1) when η_ℓ + η_h ~ 1, canceling the pole.

8.3 RG Consistency Check

The anomalous dimensions satisfy:
```
γ_h = γ^j_g + 2γ^j_q + γ^s_{qq} + γ^s_g = γ^j_g + 2γ^j_q + γ^s_{qg} + γ^s_q
```

This ensures μ-independence of the resummed result.

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9. Key Takeaways for C-Parameter Extension

9.1 What Should Carry Over

1. **Factorization structure:** Trijet hard × jet functions × soft function
2. **Six-sextant soft function decomposition** (geometry is the same)

3. **Power counting:** $r \sim \lambda$ with soft/collinear scaling
4. **Absence of NGLs** (continuity argument should apply)
5. **Hard function and jet functions** (observable-independent)

9.2 What Will Be Different

1. **Observable definition:** $C = 3(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)$ with eigenvalue structure
2. **Shoulder location:** $C_{sh} = 3/4$ instead of $1/3$
3. **Nature of discontinuity:** Step vs. kink may change resummed kernel structure
4. **Measurement constraint:** The function $W(r, m_j, k_i)$ will have different form
5. **Soft function projections:** $\bar{v}_j$ vectors may differ for C-parameter

9.3 Open Questions

1. Does the step discontinuity change the $\sin(\pi\eta)/\sin(\pi(\eta_\ell + \eta_h))$ structure?
2. Is there a left shoulder, right shoulder, or both for C-parameter?
3. Are the soft function projection vectors the same or different?
4. What is the explicit form of the measurement constraint W for C-parameter?

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10. Notation Summary

Symbol	Meaning
τ	$1 - T$ (thrust variable)
ρ	Heavy jet mass / Q^2
r	$1/3 - \rho$ (left shoulder variable for HJM)
t	$\tau - 1/3$ (right shoulder variable for thrust)
s	$\rho - 1/3$ (right shoulder variable for HJM)
s_{ij}	$(p_i + p_j)^2/Q^2$
η_ℓ, η_h	RG evolution parameters for light/heavy hemispheres
Γ_0	Leading cusp anomalous dimension = 4
$A_\Gamma(\nu,\mu)$	Integrated cusp anomalous dimension
$S(\nu,\mu)$	Sudakov RG kernel

References for Further Reading

1. **Original Sudakov shoulder paper:** Catani & Webber, hep-ph/9710333
2. **NNLL extension:** Bhattacharya et al., arXiv:2306.08033
3. **C-parameter in dijet region:** Gardi & Magnea, hep-ph/0306094
4. **SCET reviews:** Becher, Neubert; Stewart, Tackmann

Document prepared for C-parameter Sudakov shoulder research project
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