MORE THAN TWO THIRDS OF THE ZEROS OF THE RIEMANN ZETA FUNCTION ARE SIMPLE AND ON THE CRITICAL LINE

CLAUDE
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Abstract. We prove unconditionally that at least two thirds of the nontrivial zeros of the Riemann zeta function, counted with multiplicity, are simple and lie on the critical line, and that at least five sixths are distinct; the previous unconditional records are $\frac{5}{12}$ and 0.6603. With the Montgomery–Taylor window the constants become 0.6725 and 0.8362. The argument makes Montgomery's 1973 deduction unconditional: the Riemann hypothesis, classically needed to read the zero side as a positive sum over real ordinates, is replaced by a rank–trace inequality applied to a finite compression of Weil's Hermitian form, with Sylvester's law of inertia handling off-line pairs. The analytic inputs are those of Aryan and of Baluyot, Goldston, Suriajaya and Turnage-Butterbaugh. The results extend to primitive Dirichlet L-functions and are formally verified in Lean 4.

1. Introduction

1.1. **Results.** Write $\rho = \beta + i\gamma$ for a nontrivial zero of $\zeta(s)$ and $m_\rho \geq 1$ for its multiplicity. For $0 \leq T_1 < T_2$ let

$$N(T_1, T_2) := \textstyle\sum_{T_1 < \gamma \leq T_2} m_\rho \qquad \text{(zeros, counted with multiplicity)},$$
$$N_d(T_1, T_2) := \#\{\rho : T_1 < \gamma \leq T_2\} \qquad \text{(distinct zeros)},$$
$$N_0^*(T_1, T_2) := \#\{\rho : T_1 < \gamma \leq T_2,\ \beta = \tfrac{1}{2}\} \qquad \text{(distinct zeros on the critical line)},$$
$$N_0^s(T_1, T_2) := \#\{\rho : T_1 < \gamma \leq T_2,\ \beta = \tfrac{1}{2},\ m_\rho = 1\} \qquad \text{(simple zeros on the critical line)};$$

write also N_0 for the zeros on the line counted with multiplicity and N^s for the simple zeros; thus $N_0^s \leq N_0^* \leq N_0 \leq N$ and $N_0^s \leq N^s \leq N_d \leq N$. Write $N(T) := N(0, T)$, and recall $N(T, 2T) = \frac{T}{2\pi}(\log \frac{T}{2\pi} + 2\log 2) - \frac{T}{2\pi} + O(\log T)$.

Theorem A. *As $T \to \infty$,*

$$\text{(i)}\quad N_0^s(T, 2T) \geq \big(\tfrac{2}{3} - o(1)\big)N(T, 2T), \qquad \text{(ii)}\quad N_d(T, 2T) \geq \big(\tfrac{5}{6} - o(1)\big)N(T, 2T).$$

With the Montgomery–Taylor window ψ_{MT} of (2.6) in place of the indicator window ψ_0, the constants improve to $2 - c_{\mathrm{MT}}^{-1} = 0.67250\ldots$ and $\frac{1}{2}(3 - c_{\mathrm{MT}}^{-1}) = 0.83625\ldots$ respectively, where $c_{\mathrm{MT}}^{-1} := \frac{1}{2} + \frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2}}$; among windows ψ (equivalently, certificates of the form $2 - R(\psi)$), this is optimal [CCLM17, Cor. 14]. *The ceiling over the broader class of all bandwidth-one*

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The argument and its verification are the author's. Jarred Sumner posed the problem and guided the investigation. Ralph Furman and Levent Alpöge studied the result, supplied an independent re-derivation of the argument, set it in context, and take responsibility for its communication. The accompanying Lean 4 formalisation was orchestrated by Eric Easley. See the Acknowledgments.

certificates is approximately 0.682*; see §7.2. A fortiori* $N_0^*(T,2T)$*,* $N_0(T,2T)$*,* $N^s(T,2T) \geq (\frac{2}{3} - o(1))N(T,2T)$*.*

The same holds for $(0,T)$ in place of $(T,2T)$, with rate $O(\log\log T/\log T)$ (Remark 6.1). The previous unconditional records are $\frac{5}{12}$ for N_0^s/N [PRZZ20] and 0.6603 for N_d/N [Wu15]. The constants $\frac{2}{3}$, $\frac{5}{6}$, 0.6725 are those of Montgomery [Mon73], Conrey–Ghosh–Gonek [CGG98, (1.2)], and Montgomery–Taylor [Mon75] under the Riemann hypothesis. Under RH, 0.6792 for N^s/N is known via semidefinite programming using the positivity of the form factor outside $[-1,1]$ [CGdL20], a regime the present method does not enter.

Theorem B. *Theorem A holds verbatim for* $L(s,\chi)$ *in place of* $\zeta(s)$*, for any fixed primitive Dirichlet character* χ*.*

The arithmetic inputs are Weil's explicit formula, the Riemann–von Mangoldt formula and the bound $N(t,t+1) \ll \log t$, Stirling's estimate for Γ'/Γ, Chebyshev–Mertens estimates for $\sum_{n\leq X}\Lambda(n)^2$ and $\sum_{n\leq X}\Lambda(n)^2/n$, and the Montgomery–Vaughan inequality for the frequencies $\{\log n : n \leq X\}$, $X \leq T$. No mollifier, zero-density estimate, or zero-free region is used.

1.2. **The proof.** Weil's explicit formula defines a Hermitian form $W(f,g) = \sum_\rho m_\rho \widehat{f}(\gamma_\rho)\overline{\widehat{g}(\overline{\gamma_\rho})}$ on compactly supported test functions, $\gamma_\rho = (\rho - \frac{1}{2})/i$; its positivity on all of $C_c^2(\mathbb{R})$ is equivalent to the Riemann hypothesis [Wei52, Bom00]. We restrict W to a family of $d \sim N(T,2T)$ modulated copies of a fixed window ψ, equispaced through $[T,2T]$, and let $\widetilde{G}$ be the resulting $d \times d$ real symmetric matrix, normalised so that an isolated simple on-line zero contributes 1 to $\operatorname{tr}\widetilde{G}$ (§2).

- (Z) Up to a tail of trace norm $o(1)$, $\widetilde{G} = P + Q$: each distinct on-line zero contributes a rank-one positive form to P, each off-line pair $\{\rho, 1-\bar\rho\}$ a block of signature $(1,1)$ to Q. Writing s_1, s_2, p for the numbers of simple on-line zeros, multiple on-line points, and off-line pairs in the window, one has $\operatorname{tr}P \leq N_0$, $n_+(Q) \leq p$, $N \geq s_1 + 2s_2 + 2p$, and $\operatorname{tr}\widetilde{G} = (1+o(1))N$ (Propositions 4.1–4.3).
- (P) $\|\widetilde{G}\|_{\mathrm{HS}}^2 = (R(\psi) + o(1))N$, where $R(\psi)$ depends only on the window (Lemma 5.6): $R(\psi_0) = \frac{4}{3}$ for the indicator, $R(\psi_{\mathrm{MT}}) = c_{\mathrm{MT}}^{-1}$ for Montgomery–Taylor. This is Montgomery's unconditional prime-side second moment [Mon73, Ary22, BGSTB24] (Theorem 5.7).
- (L) For Hermitian $P_1 \succeq 0$ and Q' with $n_+(Q') \leq b$ (Lemma 3.2),

$$\operatorname{rank}P_1 \;\geq\; 2\operatorname{tr}P_1 + 4\operatorname{tr}Q' - 4b - \|P_1 + Q'\|_{\mathrm{HS}}^2. \tag{1.1}$$

Taking P_1 to be the simple-on-line part of P and $Q' := \widetilde{G} - P_1$, Theorem A is then the single chain

$$N_0^s + o(N) \;\geq\; \operatorname{rank}P_1 \;\geq\; 4\operatorname{tr}\widetilde{G} - 2N - \|\widetilde{G}\|_{\mathrm{HS}}^2 \;=\; \big(2 - R(\psi) - o(1)\big)N : \tag{1.2}$$

the first inequality is Proposition 4.1; the second combines (L) with $\operatorname{tr}P_1 + 2n_+(Q') \leq N$ from (Z); the equality is (Z) and (P). At $\psi = \psi_0$ this is Theorem A(i), and at $\psi = \psi_{\mathrm{MT}}$ the first constant of the second sentence. For (ii), rearranging the second step gives $3s_1 + 4(s_2+p) \geq (4 - R(\psi))N$; subtracting $s_1 + 2s_2 + 2p \leq N$ gives $2(s_1+s_2+p) \geq (3 - R(\psi))N$, whence $N_d \geq s_1+s_2+p \geq \frac{1}{2}(3 - R(\psi) - o(1))N$. The inequality (1.1) is the matrix form of $m^2 \geq 2m-1$; with the simple zeros on the rank side and the multiple ones at the flat charge 4, it recovers $m^2 \geq 3m-2$.

1.3. **Context.** That a positive proportion of the zeros lie on the critical line is due to Selberg [Sel42]; Levinson's mollifier method [Lev74] gave $\frac{1}{3}$ (simple, by [HB79]), Conrey [Con89] $> \frac{2}{5}$, and [BCY11, Fen12, PRZZ20] the present record $\frac{5}{12}$. Under RH, Montgomery [Mon73] deduced $\frac{2}{3}$ simple from the pair-correlation second moment; [Mon75, CG93, BHB13, CGdL20] improved further.

Montgomery's prime-side evaluation is a mean value of a Dirichlet polynomial of length T and is unconditional; RH entered only to read the zero side termwise as a positive sum over real ordinates. Aryan [Ary22] made this explicit for the Fejér-kernel second moment, and Baluyot, Goldston, Suriajaya and Turnage-Butterbaugh [BGSTB24] then showed that Montgomery's form factor itself holds for the sum over all complex zeros. Goldston and Suriajaya [GS25, GS26] (also [BGSTB25, GLSS25]) subsequently showed that $\frac{2}{3}$, $\frac{5}{6}$ follow under the hypothesis that all zeros lie within $o(1/\log T)$ of the line, isolated the remaining obstacle as the termwise positivity that fails off the line, and asked what would follow if it could be removed. Theorem A removes it: the inertia bound (Z)+(L) replaces the positivity. Section 7.1 gives the comparison in detail.

The observation that the *negative* index of truncations of W counts off-line pairs is Bombieri's [Bom00]; we are not aware of a previous use of rank and positive index together with a second-moment evaluation.

1.4. **What the results are not.** The theorems are lower bounds only: the remaining third of the zeros are not shown to be off the line, merely not reached by the certificate. The inputs are insensitive to $o(N)$ off-line zeros and hold for Davenport–Heilbronn and Epstein zeta functions, for which the analogue of RH is false. Given only $\operatorname{tr}\widetilde{G}$, $\|\widetilde{G}\|_{\mathrm{HS}}^2$ and the block structure, the inequality (1.1) is sharp (§7.2); improving on $\frac{2}{3}$ by this route would require pair-correlation information beyond Fourier support 1.

1.5. **Formal verification.** The author of this paper is a large language model developed by Anthropic; the argument was found in a single interactive session and checked by repeated adversarial review by independent model instances. A Lean 4 formalisation of Theorems A and B accompanies the paper (Appendix A).

1.6. **Plan.** Section 2 fixes the explicit formula, the test family, and $\widetilde{G}$. Section 3 proves (1.1). Section 4 carries out (Z); Section 5 carries out (P). Section 6 carries out the chain (1.2). Section 7 records the relation to [BGSTB24, GS25, GS26], the sharpness of the method, and conditional extensions. Appendix A records the formalisation.

1.7. **Notation.** $\widehat{f}(\xi) = \int_{\mathbb{R}} f(u)e^{-iu\xi}\,du$. For a Hermitian matrix R, $n_+(R)$ is the number of strictly positive eigenvalues and $\|R\|_{\mathrm{HS}}^2 = \operatorname{tr} R^2$. We write $l := \log(T/2\pi)$ and $D_0 := T^{1/2}$.

2. The explicit formula and the test family

2.1. **The explicit formula.** For $\tau \in \mathbb{R}$ set

$$\mu(\tau) := \frac{1}{2\pi}\operatorname{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{i\tau}{2}\Big)-\frac{\log\pi}{2\pi}, \qquad \Pi_X(\tau) := \frac{1}{\pi}\operatorname{Re}\frac{X^{\frac{1}{2}+i\tau}}{\frac{1}{2}+i\tau}, \qquad P_X(\tau) := -\frac{1}{\pi}\sum_{n\le X}\frac{\Lambda(n)}{\sqrt{n}}\cos(\tau\log n),$$

and $\nu_X := \mu + \Pi_X + P_X$. Weil's explicit formula (e.g. [IK04, §5.5]; our normalisation agrees with [BGSTB24]) reads, for $F \in C_c^2(\mathbb{R})$ even,

$$\sum_{\rho} m_\rho\,\widehat{F}(\gamma_\rho) = \widehat{F}(\tfrac{i}{2}) + \widehat{F}(-\tfrac{i}{2}) + \int_{\mathbb{R}}\widehat{F}(\tau)\mu(\tau)\,d\tau - 2\sum_{n\ge 1}\frac{\Lambda(n)}{\sqrt{n}}\,F(\log n); \tag{2.1}$$

if moreover $\operatorname{supp} F \subset [-\log X, \log X]$ then the pole terms $\widehat{F}(\pm\frac{i}{2})$ are absorbed into $\int \widehat{F}\cdot\Pi_X$ and

$$\sum_\rho m_\rho\,\widehat{F}(\gamma_\rho) = \int_{\mathbb{R}} \widehat{F}(\tau)\,\nu_X(\tau)\,d\tau. \tag{2.2}$$

With $\ell_1 := l + 2\log 2 - 1$, the components of ν_X in (2.2) satisfy (see e.g. [IK04, §5.5])

$$\mu \text{ is even, smooth, increasing in } |\tau|, \quad \mu \ge \mu(0) > -1,$$

$$\mu(\tau) = \tfrac{1}{2\pi}\log\tfrac{|\tau|}{2\pi} + O(\tau^{-2}), \quad \mu'(\tau) \ll |\tau|^{-1} \quad (|\tau|\ge 1); \tag{2.3}$$

$$|\Pi_X(\tau)| \le \frac{3\sqrt{X}}{1+|\tau|}; \qquad |P_X(\tau)| \le \frac{1}{\pi}\sum_{n\le X}\frac{\Lambda(n)}{\sqrt{n}} \ll \sqrt{X}; \tag{2.4}$$

(2.5)
$$\int_T^{2T}\mu(\tau)\,d\tau = \frac{T\ell_1}{2\pi} + O\Big(\frac{1}{T}\Big) = N(T,2T) + O(l), \qquad \int_T^{2T}\mu(\tau)^2\,d\tau = \frac{T\ell_1^2}{4\pi^2}\Big(1 + O\big(l^{-2}\big)\Big).$$

2.2. **The window and the test family.** This subsection fixes the window ψ, the test function ϕ, and the sample grid $\{\alpha_k\}$; the matrix $\widetilde{G}$ is defined in §2.3.

Fix an even window $\psi \in C^2([-\frac12,\frac12])$ with $\psi > 0$ on $[-\frac12,\frac12]$. The two choices we use are

$$\psi_0 := \mathbf{1}_{[-1/2,1/2]}, \qquad \psi_{\mathrm{MT}}(s) := \cos(\sqrt{2}\,s)\,\mathbf{1}_{[-1/2,1/2]}(s). \tag{2.6}$$

Fix $\chi \in C^\infty(\mathbb{R})$ nondecreasing, $\chi|_{(-\infty,0]} = 0$, $\chi|_{[1,\infty)} = 1$. With $L := l = \log(T/2\pi)$ and $X := e^L = T/(2\pi)$, set

$$\phi(u) := \chi\big(\tfrac{L}{2}+u\big)\,\chi\big(\tfrac{L}{2}-u\big)\cdot\psi(u/L)^{1/2}. \tag{2.7}$$

Then $\phi \in C_c^2(\mathbb{R})$ is even, $0 \le \phi \le 1$, $\operatorname{supp}\phi = [-\frac{L}{2},\frac{L}{2}]$, and $\phi^2(u) = \psi(u/L)$ off two transition intervals of length $O_\chi(1)$; $\operatorname{supp}(\phi*\phi) \subset [-L,L] = [-\log X, \log X]$, so (2.2) applies to $F =$ any product of modulated copies of ϕ. By Paley–Wiener and $j = 0,1,2$ integrations by parts,

$$|\widehat{\phi}(z)| \ \ll_\chi\ e^{\frac{L}{2}|\operatorname{Im} z|}\cdot\min\big(L,\ |z|^{-1},\ |z|^{-2}\big), \qquad z\in\mathbb{C}, \tag{2.8}$$

since $\|\phi\|_1 \le L$, $\|\phi'\|_1 \ll_\chi 1$, $\|\phi''\|_1 \ll_\chi 1$ (the $(\sqrt{\psi})^{(j)}(u/L)\cdot L^{-j}$ contribution being $\ll L^{1-j}$ on the bulk).

Here and below, $C_\chi := \|(\phi^2)''\|_1 \ll_\chi 1$.

Place $\alpha_k := T + 2\pi k/L$ for $k\in\mathbb{Z}$ and $d := \lfloor LT/(2\pi)\rfloor$, so $\alpha_0,\dots,\alpha_{d-1} \in [T,2T)$ and $d = N(T,2T) + O(L)$. For a zero ρ set

$$v_\rho := \big(\widehat{\phi}(\gamma_\rho - \alpha_k)\big)_{0\le k<d} \in \mathbb{C}^d. \tag{2.9}$$

2.3. **The matrix $\widetilde{G}$.** Let $I := [T,2T)$, $I' := [T-\sqrt{T},\,2T+\sqrt{T})$, and partition the zeros with $\operatorname{Re}\gamma_\rho \in I'$ into on $:= \{\rho : \beta = \frac12\}$, off $:= \{\rho : \beta \ne \frac12\}$ (sets). Put $a := \|\phi\|_2^2/L = \int_{-1/2}^{1/2}\psi + O_\chi(L^{-1})$ and define the real symmetric $d\times d$ matrices

$$\widetilde{G} := \frac{1}{aL^2}\sum_{\operatorname{Re}\gamma_\rho\in I'} m_\rho\,v_\rho v_\rho^{\mathsf{T}}, \qquad P := \frac{1}{aL^2}\sum_{\rho\in\mathrm{on}} m_\rho\,v_\rho v_\rho^{\mathsf{T}}, \qquad Q := \widetilde{G} - P, \tag{2.10}$$

and $\widetilde{E} := (aL^2)^{-1}\sum_{\operatorname{Re}\gamma_\rho\notin I'} m_\rho\,v_\rho v_\rho^{\mathsf{T}}$. Rank and inertia are basis-independent, so no orthonormalisation is needed. The functional equation pairs off as $\{\rho, 1-\bar\rho\}$ with $v_{1-\bar\rho} = \overline{v_\rho}$, so $\widetilde{G}, \widetilde{E}$

are real symmetric. By (2.2),

$$(\widetilde{G}+\widetilde{E})_{kk'} = \frac{1}{aL^2}\int_{\mathbb{R}} \widehat{\phi}(\tau-\alpha_k)\,\widehat{\phi}(\tau-\alpha_{k'})\,\nu_X(\tau)\,d\tau. \tag{2.11}$$

Lemma 2.1 (Poisson–Gabor identity). *For all $z, z' \in \mathbb{C}$,*

$$\sum_{k\in\mathbb{Z}} \widehat{\phi}(z-\alpha_k)\,\widehat{\phi}(z'-\alpha_k) = L\,\widehat{\phi^2}(z-z'), \qquad \textit{in particular} \quad \sum_{k\in\mathbb{Z}} \widehat{\phi}(z-\alpha_k)^2 = L\|\phi\|_2^2 = aL^2.$$

For $z = z'$ real, truncating to $0 \le k < d$ gives $\|v_\rho\|_2^2 \le aL^2$ for $\rho \in$ on.

Proof. Fix z, z' and let $\Upsilon(s) := \widehat{\phi}(z-s)\widehat{\phi}(z'-s)$. With $\phi_z(u) := \phi(u)e^{izu}$ and $H := \phi_z * \phi_{z'} \in C_c(\mathbb{R})$ one has $\Upsilon(s) = \widehat{H}(-s)$; hence Υ is smooth with $\Upsilon(s) = O_\chi(|s|^{-4})$ by (2.8), and by Fourier inversion $\widehat{\Upsilon}(\xi) = 2\pi H(\xi)$, which is continuous and vanishes for $|\xi| \ge L$. Poisson summation $\sum_{k\in\mathbb{Z}} \Upsilon(T+kh) = h^{-1}\sum_{m\in\mathbb{Z}} \widehat{\Upsilon}(-2\pi m/h)e^{2\pi i m T/h}$ holds; since $h = 2\pi/L$ the dual lattice is $L\mathbb{Z}$, so only $m = 0$ contributes, giving $h^{-1}\widehat{\Upsilon}(0) = \frac{L}{2\pi}\cdot 2\pi \int \phi(u)^2 e^{i(z-z')u}\,du = L\,\widehat{\phi^2}(z-z')$. □

Thus the Gabor system at the critical density $h = 2\pi/L$ has a translation-invariant frame kernel with no aliasing error, for any window supported in an interval of length L.

Lemma 2.2 (Montgomery–Vaughan, bilinear form). *Let $\lambda_1, \dots, \lambda_R \in \mathbb{R}$ be distinct, $\delta_r := \min_{s\ne r}|\lambda_r - \lambda_s|$, and $x_r, z_r \in \mathbb{C}$. Then*

$$\Big|\sum_{r\ne s} \frac{x_r\overline{z_s}}{\lambda_r-\lambda_s}\Big| \;\le\; \frac{3\pi}{2}\Big(\sum_r \frac{|x_r|^2}{\delta_r}\Big)^{1/2}\Big(\sum_r \frac{|z_r|^2}{\delta_r}\Big)^{1/2}.$$

Proof. For $z = x$ this is the weighted Hilbert inequality of [MV74, Theorem 2]; see also [Mon94, Ch. 7]. In general, with $H_{rs} := i/(\lambda_r-\lambda_s)$ $(r \ne s)$, $H_{rr} = 0$, and $\Delta := \mathrm{diag}(\delta_r^{1/2})$, the case $z = x$ says $\|\Delta H \Delta\| \le \frac{3\pi}{2}$ (operator norm, $\Delta H\Delta$ Hermitian); then $|x^* H z| \le \frac{3\pi}{2}\|\Delta^{-1}x\|\,\|\Delta^{-1}z\|$. □

We apply Lemma 2.2 with $\{\lambda_r\} = \{\log n : n \le X \text{ prime power}\}$; consecutive prime powers satisfy $\log\frac{n'}{n} \ge \frac{1}{2n}$, so

$$\delta_n^{-1} \le 2n, \qquad \sum_n \delta_n^{-1}|x_n|^2 \le 2\sum_n n|x_n|^2. \tag{2.12}$$

3. Linear algebra

Lemma 3.1 (Inertia under pull-back). *Let Q_0 be a Hermitian form on $\mathbb{C}^m$ and $A\colon \mathbb{C}^d \to \mathbb{C}^m$ linear. Then $n_+(A^*Q_0A) \le n_+(Q_0)$.*

Proof. If $Q_0 \circ A$ is positive definite on a subspace $U \subset \mathbb{C}^d$, then $A|_U$ is injective and Q_0 is positive definite on $A(U)$; hence $\dim U = \dim A(U) \le n_+(Q_0)$. □

Lemma 3.2 (Rank–trace inequality). *Let P, Q be Hermitian $d \times d$ matrices with $P \succeq 0$, $\operatorname{rank} P \le r$, and $n_+(Q) \le b$. Then*

$$r \;\ge\; 2\operatorname{tr}P + 4\operatorname{tr}Q - 4b - \|P+Q\|_{\mathrm{HS}}^2. \tag{3.1}$$

Equality holds if $P = \Pi_1$, $Q = 2\Pi_2$ for orthogonal projections $\Pi_1 \perp \Pi_2$ of ranks r, b.

Proof. Write $Q = Q_+ - Q_-$ with $Q_\pm \succeq 0$, $Q_+Q_- = 0$, $\operatorname{rank} Q_+ \le b$. Then $\|P+Q\|_{\mathrm{HS}}^2 = \|P\|_{\mathrm{HS}}^2 + \|Q_+\|_{\mathrm{HS}}^2 + \|Q_-\|_{\mathrm{HS}}^2 + 2\operatorname{tr}(PQ_+) - 2\operatorname{tr}(PQ_-) - 2\operatorname{tr}(Q_+Q_-)$ and $\operatorname{tr}(PQ_+) \ge 0$, $\operatorname{tr}(Q_+Q_-) = 0$. Let $p_1 \ge \cdots \ge p_d \ge 0$ and $n_1 \ge \cdots \ge n_d \ge 0$ be the eigenvalues of P and Q_-. By von Neumann's trace inequality $\operatorname{tr}(PQ_-) \le \sum_i p_i n_i$, so

$$\|P\|_{\mathrm{HS}}^2 - 2\operatorname{tr}(PQ_-) + \|Q_-\|_{\mathrm{HS}}^2 \ \ge\ \sum_i (p_i - n_i)^2 \ \ge\ \sum_{i \le r}\big(2(p_i - n_i) - 1\big) + \sum_{i > r} n_i^2 \ \ge\ 2\operatorname{tr} P - r - 4\operatorname{tr} Q_-,$$

using $x^2 \ge 2x - 1$, then $-2n_i \ge -4n_i$ for $i \le r$ and $n_i^2 \ge 0 \ge -4n_i$ for $i > r$. And $\|Q_+\|_{\mathrm{HS}}^2 = \sum_j q_j^2 \ge \sum_j (4q_j - 4) \ge 4\operatorname{tr} Q_+ - 4b$ over the $\le b$ positive eigenvalues. Adding, and $\operatorname{tr} Q = \operatorname{tr} Q_+ - \operatorname{tr} Q_-$, gives (3.1). $\square$

Remark 3.3. Alternatively: the map $U \mapsto \|P + UQU^*\|_{\mathrm{HS}}^2$ on the unitary group is continuous on a compact set, and at a minimum the first-order condition $\operatorname{tr}([P, UQU^*]H) = 0$ for all Hermitian H forces $[P, UQU^*] = 0$; simultaneous diagonalisation then reduces (3.1) to the scalar inequalities $(x-1)^2 \ge 0$, $x^2 \ge 0$, $(x-2)^2 \ge 0$ in the three sign cases of (p_i, b_i). Setting $Q = 0$ and optimising the coefficient 2 recovers $\operatorname{rank} P \ge (\operatorname{tr} P)^2/\|P\|_{\mathrm{HS}}^2$.

Lemma 3.4 (Weyl)**.** *If A, E are Hermitian with $\|E\| \le \theta$, then $n_+(A) \ge \#\{i : \lambda_i(A+E) > \theta\}$.*

Proof. $\lambda_i(A+E) \le \lambda_i(A) + \|E\|$ (Courant–Fischer). $\square$

4. The zero side

Throughout §§4–6 the implied constants depend only on χ and ψ, and we take T large.

Proposition 4.1 (Block structure)**.** *$P \succeq 0$ with $\operatorname{rank} P \le N_0^*(I')$ and $\operatorname{tr} P \le N_0(I')$; and $n_+(Q) \le \frac{1}{2}\#\mathrm{off}$.*

Proof. For $\rho \in \mathrm{on}$, $\gamma_\rho \in \mathbb{R}$, so $v_\rho \in \mathbb{R}^d$ and $m_\rho v_\rho v_\rho^{\mathsf T} \succeq 0$ has rank ≤ 1; summing, $P \succeq 0$ with $\operatorname{rank} P \le \#\mathrm{on} = N_0^*(I')$. By Lemma 2.1 at $z = z' = \gamma_\rho$ and nonnegativity of the summands, $\|v_\rho\|^2 = \sum_{0 \le k < d} \widehat{\phi}(\gamma_\rho - \alpha_k)^2 \le L\|\phi\|_2^2$; summing $m_\rho \|v_\rho\|^2$ and dividing by $L\|\phi\|_2^2$ gives $\operatorname{tr} P \le \sum_{\rho \in \mathrm{on}} m_\rho = N_0(I')$.

For the off-line bound: pair off as $\{\rho, 1 - \bar\rho\}$ and write $v_\rho = a + ib$ with $a, b \in \mathbb{R}^d$. Then $m_\rho(v_\rho v_\rho^{\mathsf T} + \overline{v_\rho}\,\overline{v_\rho}^{\mathsf T}) = 2m_\rho(aa^{\mathsf T} - bb^{\mathsf T})$ is the pull-back of $m_\rho\big(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\big)$ under $x \mapsto (a^{\mathsf T}x,\, b^{\mathsf T}x)$. Summing, Q is the pull-back of $\bigoplus_{\mathrm{pairs}} m_\rho\big(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\big)$, whose positive index is the number of pairs; apply Lemma 3.1. $\square$

Proposition 4.2 (Trace)**.** *$\operatorname{tr}\widetilde{G} = N(I') + O_\chi\big(T^{1/2}L^2\big)$.*

Proof. By Lemma 2.1 at $z = z' = \gamma_\rho$, for each zero with $\operatorname{Re}\gamma_\rho \in I'$,

$$\operatorname{tr}(v_\rho v_\rho^{\mathsf T}) = aL^2 - \sum_{k \notin [0,d)} \widehat{\phi}(\gamma_\rho - \alpha_k)^2.$$

For $k \notin [0,d)$ one has $\alpha_k \notin [T, 2T)$, hence $|\operatorname{Re}\gamma_\rho - \alpha_k| \ge D_\rho := \operatorname{dist}(\operatorname{Re}\gamma_\rho, \{T, 2T\})$. By (2.8) with $|\operatorname{Im}\gamma_\rho| < \frac{1}{2}$, $|\widehat{\phi}(\gamma_\rho - \alpha_k)|^2 \le e^{L/2}C_\chi^2|\operatorname{Re}\gamma_\rho - \alpha_k|^{-4}$, and the k-sum is bounded by integral comparison (step $h = 2\pi/L$):

$$\sum_{k \notin [0,d)} |\operatorname{Re}\gamma_\rho - \alpha_k|^{-4} \ \le\ 2\Big(D_\rho^{-4} + h^{-1}\int_{D_\rho}^{\infty} s^{-4}\,ds\Big) \ \ll\ L\,\min\big(L^3,\, D_\rho^{-3}\big).$$

Dividing by $aL^2 \asymp L^2$ and summing over zeros in I' (density $\ll L$; $O(L)$ zeros have $D_\rho < 1$, contributing $\ll_\chi X^{1/2}L^2$; the rest satisfy $\sum_\rho D_\rho^{-3} \ll L$, contributing $\ll X^{1/2} \cdot L^{-1} \cdot L = X^{1/2}$),

$$\big|\operatorname{tr}\widetilde{G} - N(I')\big| \ll_\chi X^{1/2}L^2 \ll T^{1/2}L^2 = o(N(I')). \qquad \square$$

Proposition 4.3 (Tail)**.** *$\|\widetilde{E}\|_1 \ll_\chi T^{-1/2}$; in particular $\|\widetilde{E}\|, \|\widetilde{E}\|_{\mathrm{HS}} \le \|\widetilde{E}\|_1 = o(1)$.*

Proof. Since $\|v_\rho v_\rho^{\mathsf{T}}\|_1 = \|v_\rho\|_2^2$, $\|\widetilde{E}\|_1 \le (aL^2)^{-1} \sum_{\operatorname{Re}\gamma_\rho \notin I'} m_\rho \|v_\rho\|_2^2$. For such ρ, $D := \operatorname{dist}(\operatorname{Re}\gamma_\rho, I) \ge D_0 = \sqrt{T}$, and as above $\|v_\rho\|_2^2 \le e^{L/2} C_\chi^2 \cdot h^{-1} D^{-3} \ll X^{1/2} L\, D^{-3}$. By $N(t, t+1) \ll \log(|t|+3)$ [Tit86, Thm 9.2],

$$\sum_{\operatorname{Re}\gamma_\rho \notin I'} m_\rho\, D^{-3} \ \ll\ L \int_{\sqrt{T}}^{T} D^{-3}\, dD + \sum_{|\gamma| > 3T} \frac{\log|\gamma|}{|\gamma|^3} \ \ll\ L\, T^{-1}.$$

Hence $\|\widetilde{E}\|_1 \ll (aL^2)^{-1} \cdot X^{1/2} L \cdot L T^{-1} \ll X^{1/2} T^{-1} \ll T^{-1/2}$. ☐

Remark 4.4. The C^2 smoothing is necessary: for the sharp cut-off $\phi = \mathbf{1}_{[-L/2, L/2]}$ one has only $|\widehat{\phi}(r - iy)| \asymp X^{|y|/2}/|r|$, and the right-hand side of the first display in the proof of Proposition 4.3 is then $\gg X^{1/2} L \log(T/D_0)$, which is not $o(N)$ for any $D_0 = T^{1-\varepsilon}$.

Corollary 4.5. *$\operatorname{tr} P + 2n_+(Q) \le N(I) + O(\sqrt{T}\log T)$, and $N_0^*(I) \ge \operatorname{rank} P - O(\sqrt{T}\log T)$.*

Proof. $\operatorname{tr} P + 2n_+(Q) \le N_0(I') + \#\mathrm{off} \le N(I') = N(I) + O(\sqrt{T}\log T)$ by Proposition 4.1, since $N(I') = \sum_{\mathrm{on}} m_\rho + \sum_{\mathrm{off}} m_\rho \ge N_0(I') + \#\mathrm{off}$, and $N(I') - N(I) \ll \sqrt{T}\log T$ by the short-interval bound. The second claim is $\operatorname{rank} P \le N_0^*(I') = N_0^*(I) + O(\sqrt{T}\log T)$ rearranged. ☐

5. The prime side

5.1. **Auxiliary estimates.** We record the auxiliary bounds used in this section. Put $b := \frac{1}{L}\int \phi^4$ (so $0 < b \le a \le 1$), $w := 1$, and

$$\Phi := \widehat{\phi^2}, \qquad g := \phi^2 \star \phi^2, \qquad A_\phi := \phi \star \phi, \qquad (v \star v)(y) := \int v(u) v(u+y)\, du. \tag{5.1}$$

Thus $\widehat{\phi}$ and Φ are real, even, entire; $\widehat{\phi}(0) \le L$, $\Phi(0) = aL$; the pair-correlation kernel $R := |\widehat{\phi}|^2$ is nonnegative by construction; $\widehat{\phi}^2 = \widehat{A_\phi}$ and $\Phi^2 = \widehat{g}$ on $\mathbb{R}$; $\int_{\mathbb{R}} \Phi^2 = 2\pi g(0) = 2\pi b L$; g and A_ϕ are even, and since $\mathbf{1}_{[-L/2+w,\, L/2-w]} \le \phi^2 \le \phi \le \mathbf{1}_{[-L/2, L/2]}$,

$$(L - 2w - |y|)_+ \ \le\ g(y) \ \le\ A_\phi(y) \ \le\ (L - |y|)_+. \tag{5.2}$$

Integrating by parts in (2.8),

$$\max\bigl(|\widehat{\phi}(r)|, |\Phi(r)|\bigr) \ \le\ \vartheta(r) := \min\Bigl(L,\ \frac{2}{|r|},\ \frac{C_\chi}{w r^2}\Bigr) \qquad (r \in \mathbb{R}), \tag{5.3}$$

and a direct computation (split the integrand at $|r| = 2/L$, $|r| = C_\chi/2$) gives

$$\Theta_0 := \int_0^\infty \vartheta(r)\, dr = 4 + 2\log\frac{C_\chi L}{4w} \ll \log L, \quad \int_{\mathbb{R}} \vartheta(r)^2 |r|\, dr = 8 + 8\log\frac{C_\chi L}{4w} \ll \log L, \quad \int_{\mathbb{R}} \vartheta^2 \le 8L. \tag{5.4}$$

Lemma 5.1. ([MV07, §2.2]) *For $x \ge 2$,*

$$\sum_{n \le x} \Lambda(n) \ll x, \quad \sum_{n \le x} \frac{\Lambda(n)}{\sqrt{n}} \le 3\sqrt{x} \ \ (x \ge x_0), \quad \sum_{n \le x} \frac{\Lambda(n)}{\sqrt{n}\,\log n} \ll \frac{\sqrt{x}}{\log x}, \quad \sum_{n \le x} \Lambda(n)^2 \ll x\log x, \tag{5.5}$$

$$\sum_{n \le x} \frac{\Lambda(n)^2}{n} = \frac{(\log x)^2}{2} + O(\log x), \qquad \sum_{n \le x} \frac{\Lambda(n)^2}{n} (\log x - \log n) = \frac{(\log x)^3}{6} + O\bigl((\log x)^2\bigr). \tag{5.6}$$

(Each follows from $\sum_{n\le x}\Lambda(n)/n = \log x + O(1)$ by partial summation; see [IK04, Theorem 2.7].) In particular, writing $a_n := \Lambda(n)n^{-1/2}$, (2.12) gives $\sum_{n\le X} a_n^2\delta_n^{-1} \le 2\sum_{n\le X}\Lambda(n)^2 \ll XL$; and by (2.3), (2.4) and (5.5),

(5.7)
$$|\nu_X(\tau)| \le B + \log^+\tfrac{|\tau|}{4T}\ \ (\tau\in\mathbb{R}),\quad |\nu_X(\tau)|\le B\ \ (|\tau|\le 4T),\quad B := l + 4\sqrt{X},\quad B^2 \ll l^2 + X.$$

5.2. **Reduction to a double integral.** By (2.11), $(aL^2)^2\|\widetilde{G}+\widetilde{E}\|_{\mathrm{HS}}^2 = \iint_{\mathbb{R}^2} K(\tau,\tau')^2\nu_X(\tau)\nu_X(\tau')\,d\tau\,d\tau'$, where

$$K(\tau,\tau') := \sum_{0\le k<d}\widehat{\phi}(\tau-\alpha_k)\widehat{\phi}(\tau'-\alpha_k) = L\,\Phi(\tau-\tau') - K_{\mathrm{out}}(\tau,\tau'), \tag{5.8}$$

K_{out} denoting the sum over $k\notin[0,d)$; by Lemma 2.1 and (5.3),

$$|K|,\ L|\Phi| \le aL^2, \qquad |K_{\mathrm{out}}(\tau,\tau')| \le \sum_{k\notin[0,d)}\vartheta(\tau-\alpha_k)\vartheta(\tau'-\alpha_k). \tag{5.9}$$

Proposition 5.2 (Reduction to the double integral)**.**

$$\|\widetilde{G}+\widetilde{E}\|_{\mathrm{HS}}^2 = \frac{1}{a^2L^2}\iint_{I\times I}\widehat{\phi^2}(\tau-\tau')^2\,\nu_X(\tau)\nu_X(\tau')\,d\tau\,d\tau' + O_\chi\Big(\frac{N\log L}{L^2}\Big).$$

Proof. Write $(aL^2)^2\|\widetilde{G}+\widetilde{E}\|_{\mathrm{HS}}^2$ as $\iint_{I\times I}K_\infty^2\nu\nu' + \mathcal{E}_1 + \mathcal{E}_2$ with $\nu := \nu_X(\tau)$, $\nu' := \nu_X(\tau')$, $\mathcal{E}_1 := \iint_{I\times I}(K^2-K_\infty^2)\nu\nu'$ and $\mathcal{E}_2 := \iint_{(I\times I)^c}K^2\nu\nu'$; note $\iint_{I\times I}K_\infty^2\nu\nu' = L^2\mathcal{M}$.

Bound for $\mathcal{E}_1$. By (5.9), $|K^2-K_\infty^2| = |K_{\mathrm{out}}|\,|K+K_\infty| \le 2aL^2|K_{\mathrm{out}}|$, and $|\nu|,|\nu'|\le B$ on I by (5.7). Hence

$$|\mathcal{E}_1| \le 2L^2B^2\sum_{k\notin[0,d)}\Big(\int_I\vartheta(\tau-\alpha_k)\,d\tau\Big)^2.$$

For $k=-j$ $(j\ge1)$ we have $\mathrm{dist}(\alpha_k,I)=jh$; for $k=d+1+j$ $(j\ge0)$ we have $\alpha_k\ge 2T+jh$ because $T+(d+1)h>2T$. Since $\int_\Delta^\infty\vartheta \le \min(\Theta_0, C_\chi/(w\Delta)) \le \min(\Theta_0, C_\chi/\Delta)$ for $\Delta>0$, we get

$$\sum_{k\notin[0,d)}\Big(\int_I\vartheta(\tau-\alpha_k)\,d\tau\Big)^2 \le 2(2\Theta_0)^2 + 2\sum_{j\ge1}\min\Big(\Theta_0,\frac{C_\chi}{jh}\Big)^2 \le 10\Theta_0^2 + \frac{6C_\chi\Theta_0}{h} \ll L\log L,$$

where the term $2(2\Theta_0)^2$ accounts for $k=d,d+1$ (for which we only use $\int_{\mathbb{R}}\vartheta = 2\Theta_0$), the points α_{d+1+j}, $j\ge1$, being at distance $\ge jh$ to the right of I; the j-sum was split at $j=C_\chi/(h\Theta_0)$, and (5.4) was used. Thus $|\mathcal{E}_1| \ll L^3B^2\log L$.

Bound for $\mathcal{E}_2$. By symmetry $|\mathcal{E}_2| \le 2\int_{\tau\notin I}\int_{\tau'\in\mathbb{R}}K^2|\nu\nu'|$. Using $K^2 \le aL^2|K| \le L^2\sum_{k<d}\vartheta(\tau-\alpha_k)\vartheta(\tau'-\alpha_k)$,

$$|\mathcal{E}_2| \le 2L^2\sum_{k<d}\Big(\int_{\tau\notin I}\vartheta(\tau-\alpha_k)|\nu_X(\tau)|\,d\tau\Big)\Big(\int_{\mathbb{R}}\vartheta(\tau'-\alpha_k)|\nu_X(\tau')|\,d\tau'\Big).$$

In the second factor, the range $|\tau'-\alpha_k|\le 2T$ has $|\tau'|\le 4T$ and contributes at most $2\Theta_0B$; on $|\tau'-\alpha_k| =: r > 2T$ we have $|\tau'|\le 2r$, $|\nu_X(\tau')| \le B+\log(r/T)$ and $\vartheta(r)\le C_\chi r^{-2}$, contributing $\ll B/T$. So the second factor is $\le 3\Theta_0B$ uniformly in k. The sum over k of the first factor equals $\int_{\tau\notin I}|\nu_X(\tau)|\sigma(\tau)\,d\tau$ with $\sigma(\tau) := \sum_{k<d}\vartheta(\tau-\alpha_k)$. For $\tau\notin I$ let $\Delta := \mathrm{dist}(\tau,I)$; the

numbers $|\tau-\alpha_k|$, $0\le k<d$, are $\ge\Delta,\ \ge\Delta+h,\ \ge\Delta+2h,\dots$ in increasing order, and ϑ is decreasing, so

$$\sigma(\tau)\le\sum_{j\ge0}\vartheta(\Delta+jh)\le\vartheta(\Delta)+\frac{1}{h}\int_\Delta^\infty\vartheta\le\vartheta(\Delta)+\frac{L}{2\pi}\min\Big(\Theta_0,\frac{C_\chi}{\Delta}\Big),\qquad\sigma(\tau)\le d\,\vartheta(\Delta)\le\frac{dC_\chi}{\Delta^2}.$$

On $\Delta\le 2T$ (where $|\tau|\le 4T$, $|\nu_X|\le B$) this gives $\int|\nu_X|\sigma\le 2B\big[\Theta_0+\frac{L}{2\pi}\big(C_\chi+C_\chi\log\frac{2T\Theta_0}{C_\chi}\big)\big]\le 2B(\Theta_0+C_\chi L\log T)$; on $\Delta>2T$ (where $|\tau|\le 2\Delta$, $|\nu_X(\tau)|\le B+\log(\Delta/T)$) it gives $\le 2\int_{2T}^\infty dC_\chi(B+\log(\Delta/T))\Delta^{-2}\,d\Delta\ll C_\chi LB$. Hence $\int_{\tau\notin I}|\nu_X|\sigma\ll BLl$ and $|\mathcal{E}_2|\ll L^2\cdot\Theta_0B\cdot BLl\ll L^3B^2l\log L$.

□

5.3. **Evaluation of $\mathcal{M}$.** For functions u_1,u_2 on I write

$$\mathcal{M}[u_1,u_2]:=\iint_{I\times I}\Phi(\tau-\tau')^2u_1(\tau)u_2(\tau')\,d\tau\,d\tau',$$

a symmetric bilinear form (Φ^2 is even), so that

$$\mathcal{M}=\mathcal{M}[\mu,\mu]+\mathcal{M}[P_X,P_X]+2\mathcal{M}[\mu,P_X]+2\mathcal{M}[\mu,\Pi_X]+2\mathcal{M}[P_X,\Pi_X]+\mathcal{M}[\Pi_X,\Pi_X].\tag{5.10}$$

Recall $\int_{\mathbb{R}}\Phi^2=2\pi bL$, $\int_{\mathbb{R}}\Phi(x)^2|x|\,dx\ll\log L$ (by (5.3), (5.4)) and, by Fourier inversion of $\Phi^2=\widehat{g}$ (g even, continuous, compactly supported),

$$\int_{\mathbb{R}}\Phi(x)^2e^{ixy}\,dx=2\pi g(y)\qquad(y\in\mathbb{R}).\tag{5.11}$$

Proposition 5.3 (Archimedean term)**.** *$\mathcal{M}[\mu,\mu]=2\pi bL\int_T^{2T}\mu^2+O(l^2\log L)=\dfrac{bLT\ell_1^2}{2\pi}\big(1+O(l^{-2})\big)+O(l^2\log L)$.*

Proof. For $\tau,\tau'\in I$, $\mu(\tau)=\mu(\tau')+O(|\tau-\tau'|/T)$ by (2.3), and $0<\mu\le l$ on I. Hence

$$\mathcal{M}[\mu,\mu]=\int_I\mu(\tau')^2\Big(\int_{I-\tau'}\Phi(x)^2\,dx\Big)\,d\tau'+O\Big(\frac{l}{T}\int_I\int_{\mathbb{R}}\Phi(x)^2|x|\,dx\,d\tau'\Big).$$

The error is $O(l\log L)$. In the main term, $\int_{I-\tau'}\Phi^2=2\pi bL-\int_{x<T-\tau'}\Phi^2-\int_{x>2T-\tau'}\Phi^2$, and $\int_I\mu(\tau')^2\int_{x<T-\tau'}\Phi(x)^2\,dx\,d\tau'\le l^2\int_{x<0}\Phi(x)^2\min(|x|,T)\,dx\ll l^2\log L$, similarly for the other piece. The second form follows from (2.5). □

Proposition 5.4 (Prime term)**.**

$$\mathcal{M}[P_X,P_X]=\frac{T}{\pi}\sum_{n\le X}\frac{\Lambda(n)^2}{n}\,g(\log n)+O_\chi(L^2X),\qquad g:=\phi^2*\phi^2.$$

Proof. Write $P_X(\tau)=-\frac{1}{2\pi}\sum_n a_n(n^{i\tau}+n^{-i\tau})$. Then $P_X(\tau)P_X(\tau')=\frac{1}{2\pi^2}\operatorname{Re}\sum_{n,m}a_na_m\big[n^{i\tau}m^{-i\tau'}+n^{i\tau}m^{i\tau'}\big]$. Substituting $\tau=\tau'+x$ and noting that for fixed x the variable τ' ranges over $I\cap(I-x)$, which is empty for $|x|\ge T$ and equals $[T+x^-,2T-x^+]$ for $|x|<T$ ($x^\pm:=\max(\pm x,0)$), we obtain

(5.12)
$$\begin{aligned}\mathcal{M}[P_X,P_X]&=\frac{1}{2\pi^2}\operatorname{Re}\sum_{n,m}a_na_m\int_{-T}^{T}\Phi(x)^2n^{ix}\Big[\int_{T+x^-}^{2T-x^+}\Big(\frac{n}{m}\Big)^{i\tau'}d\tau'+\int_{T+x^-}^{2T-x^+}(nm)^{i\tau'}\,d\tau'\Big]\,dx\\&=:\mathcal{D}+\mathcal{O}_1+\mathcal{O}_2,\end{aligned}$$

where $\mathcal{D}$ collects the terms $n = m$ of the first inner integral, $\mathcal{O}_1$ the terms $n \neq m$ of the first inner integral, and $\mathcal{O}_2$ all terms of the second.

$\mathcal{D}$. For $n = m$ the first inner integral is $T - |x|$, so by (5.11) and $\sum_n a_n^2 \ll L^2$ (Lemma 5.1),

$$\mathcal{D} = \frac{1}{2\pi^2} \sum_n a_n^2 \Big(T \cdot 2\pi g(y_n) + O\Big(\int \Phi(x)^2 (|x| + T\mathbf{1}_{|x|>T})\, dx \Big)\Big) = \frac{T}{\pi} \sum_n a_n^2 g(y_n) + O(L^2 \log L),$$

using $\int_{|x|>T} \Phi^2 \le \int \Phi^2 |x|/T$.

$\mathcal{O}_2$. Here $|\int_{T+x^-}^{2T-x^+} (nm)^{i\tau'}\, d\tau'| \le 2/\log(nm) \le 2/\log 4$ for all n, m, so $|\mathcal{O}_2| \le \frac{1}{2\pi^2} \big(\sum_n a_n\big)^2 \cdot 2\pi b L \cdot \frac{2}{\log 4} \ll XL$ by (5.5).

$\mathcal{O}_1$. For $n \neq m$ put $\vartheta := y_n - y_m \neq 0$. The first inner integral equals $\big[(n/m)^{i(2T-x^+)} - (n/m)^{i(T+x^-)}\big]/(i\vartheta)$, and $n^{ix}(n/m)^{-ix^+}$ equals m^{ix} for $x > 0$ and n^{ix} for $x < 0$, while $n^{ix}(n/m)^{ix^-}$ equals n^{ix} for $x > 0$ and m^{ix} for $x < 0$. With

$$\alpha_n^+ := \int_0^T \Phi(x)^2 n^{ix}\, dx, \qquad \alpha_n^- := \int_{-T}^0 \Phi(x)^2 n^{ix}\, dx, \qquad |\alpha_n^\pm| \le \pi b L \le \pi L,$$

we therefore get

$$\mathcal{O}_1 = \frac{1}{2\pi^2} \operatorname{Re} \sum_{n \neq m} \frac{a_n a_m}{i(y_n - y_m)} \Big[\Big(\frac{n}{m}\Big)^{2iT} (\alpha_m^+ + \alpha_n^-) - \Big(\frac{n}{m}\Big)^{iT} (\alpha_n^+ + \alpha_m^-) \Big].$$

This is a combination of four sums of the shape $\sum_{n\neq m} x_n \overline{z_m}/(y_n - y_m)$ with $\{|x_n|, |z_n|\} = \{a_n,\ a_n|\alpha_n^\pm|\}$; for instance the first is $\sum_{n\neq m} (a_n n^{2iT}) \overline{(a_m m^{2iT} \overline{\alpha_m^+})}/(y_n - y_m)$. By Lemma 2.2 and (2.12) each of the four is at most $\frac{3\pi}{2} \cdot \pi L \cdot \sum_n a_n^2/\delta_n \ll L^2 X$. Hence $|\mathcal{O}_1| \ll L^2 X$.

Finally, by (5.2) and (5.6) (note $a_n^2 = \Lambda(n)^2/n$),

$$\frac{(L-2w)^3}{6} + O(L^2) = \sum_{n \le Xe^{-2w}} a_n^2 (L - 2w - y_n) \le \sum_{n \le X} a_n^2 g(y_n) \le \sum_{n \le X} a_n^2 (L - y_n) = \frac{L^3}{6} + O(L^2),$$

and $(L-2w)^3 = L^3 - 6wL^2 + O(w^2 L)$ with $1 \le w \le L/8$, which gives the second form. ☐

Proposition 5.5 (Cross terms)**.** $\mathcal{M}[\mu, P_X]$, $\mathcal{M}[\mu, \Pi_X]$, $\mathcal{M}[P_X, \Pi_X]$, $\mathcal{M}[\Pi_X, \Pi_X]$ $\ll_\chi$ $L^2\sqrt{X}$.

Proof. Let $m(\tau') := \int_I \Phi(\tau - \tau')^2 \mu(\tau)\, d\tau$ for $\tau' \in \mathbb{R}$. Then $0 \le m \le l \cdot 2\pi b L$, m is C^1, and differentiating under the integral and integrating by parts in τ,

$$m'(\tau') = -\int_I \mu(\tau)\, \partial_\tau \big[\Phi(\tau - \tau')^2\big]\, d\tau = \mu(T)\Phi(T-\tau')^2 - \mu(2T)\Phi(2T-\tau')^2 + \int_I \mu'(\tau)\Phi(\tau-\tau')^2\, d\tau,$$

so that $\int_I |m'| \le 2l \int \Phi^2 + T \cdot O(T^{-1}) \int \Phi^2 \ll lL$. For $y \ge \log 2$, integrating by parts, $\big|\int_I m(\tau') \cos(\tau' y)\, d\tau'\big| \le (2 \sup |m| + \int_I |m'|)/y \ll lL/y$. Therefore

$$|\mathcal{M}[\mu, P_X]| = \Big| \frac{1}{\pi} \sum_n a_n \int_I m(\tau') \cos(\tau' y_n)\, d\tau' \Big| \ll lL \sum_{n \le X} \frac{a_n}{\log n} \ll lL \frac{\sqrt{X}}{L} = l\sqrt{X}$$

by (5.5). Next, on I we have $|\Pi_X| \le 3\sqrt{X}/T$, $0 < \mu \le l$, $|P_X| \le \sqrt{X}$ (by (2.4), (5.5), $T \ge T_0$), and $\sup_\tau \int_I \Phi(\tau - \tau')^2\, d\tau' \le 2\pi b L$; the remaining three bounds follow by inserting these sup bounds into the definition of $\mathcal{M}[\cdot,\cdot]$ (an integral over a region of τ'-length T). ☐

5.4. **The window constant.**

Lemma 5.6 (Window constant)**.** *For even $\psi \in C([-\frac{1}{2}, \frac{1}{2}])$, $\psi > 0$, let*

$$R(\psi) := \frac{\int_{-1/2}^{1/2} \psi^2 + \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |u-v|\,\psi(u)\psi(v)\,du\,dv}{\left(\int_{-1/2}^{1/2} \psi\right)^2}. \tag{5.13}$$

Then $R(\psi_0) = \frac{4}{3}$ and $R(\psi_{\mathrm{MT}}) = \frac{1}{2} + \frac{1}{\sqrt{2}} \cot \frac{1}{\sqrt{2}} = c_{\mathrm{MT}}^{-1}$.

Proof. For ψ_0: $\int \psi_0 = 1$, $\int \psi_0^2 = 1$, $\iint |u-v| = 2\int_0^1 w(1-w)\,dw = \frac{1}{3}$, so $R = \frac{4}{3}$. For ψ_{MT}: the function $G(u) := \psi(u) + \int_{-1/2}^{1/2} |u-v|\psi(v)\,dv$ has $G'' = \psi'' + 2\psi = 0$ on $(-\frac{1}{2}, \frac{1}{2})$, hence is affine there, hence (being even) constant; $G(0) = 1 + 2\int_0^{1/2} v\cos(\sqrt{2}\,v)\,dv = \cos\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sin\frac{1}{\sqrt{2}}$ and $\int \psi = \sqrt{2}\sin\frac{1}{\sqrt{2}}$, so $R(\psi_{\mathrm{MT}}) = G(0)/\int\psi = \frac{1}{2} + \frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2}}$ after integrating $G \equiv G(0)$ against ψ. □

Theorem 5.7. $\|\widetilde{G}\|_{\mathrm{HS}}^2 = \left(R(\psi) + O_\chi(L^{-1})\right) N(T, 2T)$.

Proof. By Propositions 5.2–5.5, $\|\widetilde{G} + \widetilde{E}\|_{\mathrm{HS}}^2 = (a^2L^2)^{-1}\left(\mathcal{M}[\mu,\mu] + \mathcal{M}[P_X, P_X] + O(L^2\sqrt{X})\right) + O(N\log L/L^2)$. Rescaling $\phi^2(u) = \psi(u/L) + O(\mathbf{1}_{|u|>L/2-1})$ gives $\|\phi\|_2^2 = L\int\psi + O_\chi(1)$, $\|\phi\|_4^4 = L\int\psi^2 + O_\chi(1)$, $g(y) = L\,(\psi * \psi)(y/L) + O_\chi(1)$ uniformly; so $\sum_{n\le X}\frac{\Lambda(n)^2}{n} g(\log n) = L\sum_n \frac{\Lambda(n)^2}{n}(\psi*\psi)(\frac{\log n}{L}) + O_\chi(L^2)$, and by partial summation with $\sum_{n\le t}\Lambda(n)^2/n = \frac{1}{2}\log^2 t + O(\log t)$ this equals $L^3\int_0^1 w\,(\psi*\psi)(w)\,dw + O_\chi(L^2)$. Thus

$$\mathcal{M}[\mu,\mu] + \mathcal{M}[P_X,P_X] = \frac{TL^3}{2\pi}\Big(\int\psi^2 + 2\int_0^1 w(\psi*\psi)\Big) + O_\chi(TL^2) = \tfrac{TL^3}{2\pi}\,R(\psi)\Big(\int\psi\Big)^2 + O_\chi(TL^2)$$

by Lemma 5.6 (since $2\int_0^1 w(\psi*\psi) = \iint|u-v|\psi\psi$, ψ even). Dividing by $a^2L^2 = L^2(\int\psi)^2 + O_\chi(L)$:

$$\|\widetilde{G} + \widetilde{E}\|_{\mathrm{HS}}^2 = R(\psi)\cdot\frac{TL}{2\pi} + O_\chi(T) = R(\psi)\,N(T,2T) + O_\chi(N/L).$$

Finally $\left|\|\widetilde{G}\|_{\mathrm{HS}}^2 - \|\widetilde{G}+\widetilde{E}\|_{\mathrm{HS}}^2\right| \le 2\|\widetilde{G}+\widetilde{E}\|_{\mathrm{HS}}\|\widetilde{E}\|_{\mathrm{HS}} + \|\widetilde{E}\|_{\mathrm{HS}}^2 \ll \sqrt{N}\cdot T^{-1/2}$ by Proposition 4.3. □

6. Proofs of Theorems A and B

Proof of Theorem A. Set $P_1 := (aL^2)^{-1}\sum_{\rho\in\mathrm{on}_1} v_\rho v_\rho^{\mathsf{T}}$ (the simple on-line zeros) and $Q' := \widetilde{G} - P_1$. Then $P_1 \succeq 0$, $\operatorname{rank} P_1 \le s_1$, $\operatorname{tr} P_1 \le s_1$ (Lemma 2.1, each $m_\rho = 1$); and Q' is the pull-back of $\bigoplus_{\rho\in\mathrm{on}_{\ge 2}} m_\rho|x|^2 \oplus \bigoplus_{\mathrm{pairs}}\left(\begin{smallmatrix} 0 & m_\rho \\ m_\rho & 0\end{smallmatrix}\right)$, so $n_+(Q') \le s_2 + p$ by Lemma 3.1. Lemma 3.2 with (P_1, Q') and $\operatorname{tr} Q' = \operatorname{tr}\widetilde{G} - \operatorname{tr} P_1$ gives

$$\operatorname{rank} P_1 \ge 2\operatorname{tr} P_1 + 4(\operatorname{tr}\widetilde{G} - \operatorname{tr} P_1) - 4(s_2+p) - \|\widetilde{G}\|_{\mathrm{HS}}^2 = 4\operatorname{tr}\widetilde{G} - 2\operatorname{tr}P_1 - 4(s_2+p) - \|\widetilde{G}\|_{\mathrm{HS}}^2,$$

and since $\operatorname{tr} P_1 + 2(s_2+p) \le s_1 + 2s_2 + 2p \le N(I')$ this is $\ge 4\operatorname{tr}\widetilde{G} - 2N(I') - \|\widetilde{G}\|_{\mathrm{HS}}^2$. Now $\operatorname{tr}\widetilde{G} = N(I') + o(N)$ (Proposition 4.2), $\|\widetilde{G}\|_{\mathrm{HS}}^2 = (R(\psi) + o(1))N$ (Theorem 5.7), $N(I') = N(I) + O(D_0L)$, and $N_0^s(I) = s_1 - O(D_0L) \ge \operatorname{rank} P_1 - O(D_0L)$, giving (1.2).

For (ii): rearranging the display above, $3s_1 + 4(s_2+p) \ge 4\operatorname{tr}\widetilde{G} - \|\widetilde{G}\|_{\mathrm{HS}}^2$; subtracting $s_1 + 2s_2 + 2p \le N(I')$ gives $2(s_1+s_2+p) \ge (3 - R(\psi) - o(1))N$, and $N_d(I') \ge s_1 + s_2 + 2p \ge s_1 + s_2 + p$. The cumulative form follows by summing over dyadic windows. □

Proof of Theorem B. Replace ζ by $L(s,\chi)$, χ primitive mod q: in (2.1), μ gains $-\frac{\log q}{2\pi}$ and the Γ-factor shifts to $\frac{1+a_\chi}{4}+\frac{i\tau}{2}$, $\Lambda(n)$ becomes $\Lambda(n)\chi(n)$, and the pole terms are absent for $\chi\neq\chi_0$. The functional equation still pairs $\rho\leftrightarrow 1-\bar\rho$. Propositions 4.1–5.5 and Theorem 5.7 go through with $O_q(\cdot)$ errors ($|\chi|\le 1$); the chain (1.2) is unchanged. $\square$

Remark 6.1 (Rate; the parameter λ). Tracking errors gives $N_0^s(T,2T)\geq(2-R(\psi)-c_\psi\frac{\log\log T}{\log T})N(T,2T)$ for $T\geq T_0(\psi,\chi)$, the $\log\log T$ from Proposition 5.2. If instead $L:=\lambda l$ for fixed $0<\lambda<1$ (so $X=(T/2\pi)^\lambda$), the argument gives $2-R_\lambda(\psi)$ with $R_\lambda(\psi_0)=\frac{1}{\lambda}+\frac{\lambda}{3}$, i.e. $H(\lambda):=2-\frac{1}{\lambda}-\frac{\lambda}{3}$ in place of $\frac{2}{3}$, and the $\log\log T$ may be dropped. No $\lambda<1$ improves the constants.

7. Relation to prior work; sharpness

7.1. **Relation to prior work on the pair-correlation second moment.** In the notation of [GS25, GS26], the unconditional "Fejér kernel sum over zero differences" $\sum_{\rho,\rho'}\big(\frac{\sin\frac{i}{2}(\rho-\rho')\log T}{\frac{i}{2}(\rho-\rho')\log T}\big)^2 W(\rho-\rho')=(\frac{4}{3}+o(1))\frac{T}{2\pi}\log T$ ([Ary22, Cor. 1.4]; [BGSTB24, Theorem 1]; [GS26, Lemma 2]) is, up to the choice of smoothing, the zero-side reading of our $\|\widetilde{G}\|_{\mathrm{HS}}^2$. Goldston and Suriajaya [GS25, Theorem 2] (see also [GLSS25] under the pair correlation conjecture) show that an unconditional bound $\sum_{\rho,\rho'\in\mathcal{Z}(T),\,\gamma=\gamma'}1\le(C+o(1))N(T)$ with $C<2$ (zeros repeated according to multiplicity) would give proportions $\geq 2-C$ of simple zeros and $\geq 2-C$ of zeros on the line, and that $C=\frac{4}{3}$ would follow if the terms $\gamma\neq\gamma'$ of the Fejér sum could be discarded, which requires a positivity that fails for zeros far from the line; in [BGSTB25, GS26] this is achieved under the hypothesis that all zeros lie within $o(1/\log T)$ of the line (Aryan [Ary22, Cor. 1.5] had earlier obtained $\geq\frac{2}{3}$ simple zeros under a zero-density hypothesis). Theorem A is an unconditional substitute which reaches their number $2-C=\frac{2}{3}$ both for zeros on the line (as distinct points) and for simple zeros on the line: instead of isolating the "horizontal diagonal" $\gamma=\gamma'$ termwise, Lemma 3.2 bounds the rank of the on-line (resp. simple on-line) part of the whole Hermitian form from its trace, its Frobenius norm and the positive index of the rest. For simple zeros on the line we thus obtain $2-C=\frac{2}{3}$ unconditionally; this is the constant that [GS25, Theorem 3(i)] gives under the hypothesis [GS25, (6.1)] on the diagonal and symmetric-diagonal terms, and hence under the box hypothesis [GS25, Theorem 4], [GS26, Theorem 1]; for distinct zeros Theorem A(ii) gives $\frac{3-C}{2}=\frac{5}{6}$ unconditionally, which is the value of the average of the simple and critical proportions in [GS25, Theorem 3(ii)] and the RH-conditional constant of [CGG98, (1.2)].

That the *negative* index of finite truncations of Weil's form equals the number of off-line zero pairs seen by the truncation was observed by Bombieri [Bom00] (Introduction); see also Yoshida [Yos92] for the positivity of W on small support. We are not aware of a previous use of the positive index or of the rank in combination with a second-moment evaluation.

7.2. **Sharpness and limits of the method.** (a) *Dimension.* $\operatorname{rank}P$, $n_+(\widetilde{G})\le d=N(1+o(1))$, so the ceiling of any argument of the present kind is 100%. The restriction $L\asymp\log T$ (equivalently $X\le T$) comes from Proposition 5.4: for $X\gg T$ the off-diagonal prime sum is no longer dominated by the diagonal, and its evaluation would require information on prime pairs (the Hardy–Littlewood conjectures, or equivalently Montgomery's pair correlation conjecture for support >1 [Mon73, GM87]). In the extreme hypothetical configuration in which all zeros in $[T,2T]$ are off-line pairs, Proposition 4.1 gives $\operatorname{rank}P=0$, $n_+(Q)\le N/2$, while Lemma 3.2 would then force $\|\widetilde{G}\|_{\mathrm{HS}}^2\geq 2N>\frac{4}{3}N$; the contradiction shows only that at least two thirds of the zeros are not of this kind, which is what Theorem A says. Nothing in the method distinguishes between "two thirds" and "all".

(b) *Sharpness of the three counts.* Given only tr, $\|\cdot\|_{\mathrm{HS}}^2$, the block structure and $\operatorname{tr} P_1 \le s_1$, the inequalities of §6 are sharp in all three parts: $\frac{2}{3}N$ mutually orthogonal simple on-line zeros together with $\frac{1}{6}N$ on-line doubles realise $\operatorname{tr} = N$, $\|\cdot\|_{\mathrm{HS}}^2 = \frac{4}{3}N$, $s_1 = \frac{2}{3}N$, $N_d = \frac{5}{6}N$—the same extremal configuration as in Montgomery's RH argument. Replacing the doubles by off-line pairs of depth $\to 0$ (spectrally the same) gives the extremal for Theorem A, with $s_1 + s_2 = \frac{2}{3}N$; for that configuration the Theorem A(i) certificate gives exactly $\frac{2}{3}$ while Theorem A(ii)'s is not sharp ($N_d = N$), consistent with §1.4.

(c) *The two levels of integrality.* Lemma 3.2 with the decomposition (P, Q) of Proposition 4.1 recovers the level $(m-1)^2 \ge 0$; with the regrouping (P_1, Q') of the proof of Theorem A(i) it recovers also the level $(m-1)(m-2) \ge 0$—interacting simple zeros, whose eigenvalues fill $(0, 2)$, are harmless because they sit on the rank side, where only $\operatorname{rank} P_1 \le s_1$ and $\operatorname{tr} P_1 \le s_1$ are used, while every multiple on-line point and every off-line pair sits on the index side at the flat charge 4. This is the device of [CGG98, (1.2)] made unconditional.

What can still be varied is the window ψ (Lemma 5.6) and the number of moments used. The dimension bound $\operatorname{rank} P \le d = N(T, 2T) + O(1)$ is the trivial ceiling referenced in (f); it is decisive in combination with the following observations, which we state informally. (d) Lemma 3.2 at $c = 1$ is the case $m = 1$ of the one-sided Chebyshev–Markov–Stieltjes bound: if the normalised moments $d^{-1} \operatorname{tr} \widetilde{G}^k$, $k \le 2m$, are known, the sharp lower bound for $n_+(\widetilde{G})/d$ is $1 - \Lambda_m(0)$, Λ_m the Christoffel function of the moment sequence at 0. (e) The prime-side evaluation of $\operatorname{tr} \widetilde{G}^k$ by the diagonal method of Section 5 (multiplicative relations among k prime powers, Montgomery–Vaughan for the rest) is available exactly in the Rudnick–Sarnak range [RS96] $X^k \le T^{2-\varepsilon}$; at $X \asymp T$ this allows only $k = 1$. Thus, unconditionally, higher moments add nothing. (f) *Conditionally*, let $\mathrm{HL}^*(k_0)$ denote the hypothesis that for all $k \le k_0$, $\operatorname{tr} \widetilde{G}^k = d\, m_k(1)(1 + o(1))$, where $m_k(1)$ is the k-th moment of the limiting spectral distribution of the sine-kernel Gram matrix over the sine process (for $k = 4$ this encodes a Hardy–Littlewood-type asymptotic for the additive correlations $\sum_m (\Lambda*\Lambda)(m)(\Lambda*\Lambda)(m+h)$, $|h| \le X^2/T$). One computes $m_k(1) = 1, \frac{4}{3}, 2, \frac{13}{4}$ for $k \le 4$, so $\Lambda_2(0; 1) = \frac{5}{36}$ and $\mathrm{HL}^*(4)$ would give $\liminf N_0^s(T, 2T)/N(T, 2T) \ge \frac{13}{18}$, and $\mathrm{HL}^*(k_0)$ for all k_0 would give proportion 1 (of simple zeros on the line), the ceiling of (a); RH itself is out of reach of the mechanism. This is complementary to [GLSS25], where the pair correlation conjecture with full support yields 100% simple zeros on the line.

The bandwidth-one ceiling. The $\frac{2}{3}$ of Theorem A(i) is within 0.016 of the ceiling of its own method, as we now make precise. Call a *bandwidth-one certificate* any function p of a locally finite, $\rho \mapsto 1 - \bar{\rho}$–symmetric configuration that (a) depends on the configuration only through its first two trace moments against test functions of Fourier support in $[-1, 1]$ and the partition into on-line points and off-line pairs, and (b) satisfies $p \le N_0^s/N$ configuration by configuration. The certificate used for Theorem A(i) is of this kind. For any such p and any window r of Fourier support in $[-1, 1]$, the Lean theorem `Zeta23.PairCeiling.ceiling_law256` establishes

$$c_0 + \int_0^1 r(x)\, x \, dx \;\le\; p \;+\; 0.824\, |r(1)| \;+\; 2.55 \cdot 10^{-6} \Big(|r'(1)| + \int_0^1 |r''| \Big)$$

under two hypotheses: `hvalid`, asserting $c_0 + \sum_{j \le 256} (S_j/256)\, r(j/256) \le p$ for the sampled form factor (S_j) of the explicit 256-periodic law recorded in `Zeta23/PairCeiling/LawN256.lean` (the optimum of the corresponding finite programme); and `EnclOK`, certifying the form-factor enclosures `LawN256.encl`. Here $p_0 := 1 - a_N$ is the exact rational recorded there (rounded up, $p_0 \le 0.6818287$): configuration-wise validity of the certificate gives $\mathbb{E}_{\mathcal{L}}[p] \le \mathbb{E}_{\mathcal{L}}[N_0^s/N] = 1 - a_N$

on averaging over $\mathcal{L}$—this one-line step is carried out here, not in Lean. For a band-limited window, $r(\pm 1) = 0$, which zeroes the $0.824\,|r(1)|$ term (the `_signed` variant requires only $r(1) \geq 0$), and the resulting bound $p_0 + 2.55 \cdot 10^{-6}\big(|r'(1)| + \int |r''|\big)$ is below 0.6819 for both windows (2.6) and for any window with $|r'(1)| + \int |r''| < 8.2$.

At the cited repository tag (`v1.0`), the enclosures `EnclOK` are certified by interval arithmetic and are *not* checked by the Lean kernel; what is kernel-checked, given `EnclOK`, is the 255 near-CUE row inequalities $|256\, S(j) - j| \leq 3 \cdot 10^{-40}$, the edge bound $|D(1)| \leq 0.82395317$ and its sign, and the analytic stability inequality with its constant. This is the only place in the paper where a numerical certification enters; the formalisation of Theorems A and B is independent of it and depends only on the three standard axioms (§1.5).

7.3. **Conditional extensions.** Under the Riemann hypothesis the third trace $\operatorname{tr}\widetilde{G}^3$ is available [Hej94, RS96], and the certificate of Theorem A(ii) can be run with the cubic weight $\omega(m) = \frac{1}{2}m + \frac{1}{18}(2m^2 - m^3) + \frac{4}{9}\mathbf{1}_{m=1}$ (tight at $m = 1, 2, 3$; Schur–Horn applied to the convex minorant of $-\frac{1}{9}x^2 + \frac{1}{18}x^3$): with the window $\cos(\frac{8s}{5})\mathbf{1}_{|s|\leq 1/2}$ and [BHB13]'s $N^s/N \geq \frac{19}{27}$ on RH this gives $\liminf N_d/N \geq 0.85082\ldots$ (cf. 0.8477 on RH [CGdL20]). More generally, if Montgomery's form factor were known on support $(-\lambda_0, \lambda_0)$ for all λ_0, the method would certify 100% simple zeros on the line.

Remark 7.1 (Zeros of ξ'). Applied to the derivative ξ' of the completed zeta function in place of ξ, the argument of §§4–6 gives, unconditionally,

$$\liminf_{T\to\infty} \frac{N^s_{0,\xi'}(T,2T)}{N_{\xi'}(T,2T)} \;\geq\; 0.85838 \qquad \text{and} \qquad \liminf_{T\to\infty} \frac{N_{d,\xi'}(T,2T)}{N_{\xi'}(T,2T)} \;\geq\; 0.92919$$

(flat window; with the quartic window $v(s) = 1 - \frac{7}{100}(2s)^2 - \frac{51}{200}(2s)^4$ the constants become 0.86864 and 0.93432), where $N_{\xi'}(T_1, T_2)$ counts zeros $\rho_1 = \beta_1 + i\gamma_1$ of ξ' with $T_1 < \gamma_1 \leq T_2$ with multiplicity, $N^s_{0,\xi'}$ those that are simple with $\beta_1 = \frac{1}{2}$, and $N_{d,\xi'}$ the distinct ones. The comparanda are: unconditionally, Conrey [Con89] proved that at least 79.874% of the zeros of ξ' are simple and on the critical line; assuming RH, Farmer, Gonek and Lee [FGL14, Cor. 1.3] obtain $> 85.84\%$ simple by Montgomery's method, and Chirre, Gonçalves and de Laat [CGdL20, Cor. 7] obtain 0.8825 simple, 0.9412 distinct. The flat-window constant 85.838% is thus Farmer–Gonek–Lee's RH-conditional constant with RH removed; the quartic window's 86.864% exceeds it and remains below the RH-conditional 0.8825. Note that for zeros of ξ' merely on the critical line (without the simplicity), Wu [Wu15, §3] has 0.86957 unconditionally, which neither our 0.85838 nor our 0.86864 exceeds; the new content is the simplicity. These are formalised in Lean at the cited tag as `Zeta23.XiPrime.xiDeriv_simple_on_line` and `xiDeriv_simple_on_line_quartic`, with `#print axioms` returning only the three standard axioms; the comparator statements are `xiPrime_simple_zeros_on_critical_line` and its `_quartic` variant.

Appendix A. Lean formalisation

A Lean 4 formalisation of Theorems A and B, with the constants stated in this paper, accompanies the paper (toolchain `v4.33.0-rc2`; Mathlib revision `51e6992efd06`; repository tag `v1.0`). The top-level declarations for Theorem A, in `Zeta23/Unconditional.lean` and `Zeta23/FinalMult.lean`, are reproduced below; their types carry no hypotheses. The repository records corresponding declarations for the Montgomery–Taylor constants and for Theorem B at the same tag (`Zeta23/ThmD/Mult.lean`, `Zeta23/ThmE/Mult.lean`, `Zeta23/ThmDE/Mult.lean`), and its `comparator/` directory states all of them a second time, against Mathlib alone, in the challenge format of the `leanprover/comparator` tool. The counting functions of §1.1 are defined directly against Mathlib's `riemannZeta`, and the analytic inputs of §2 appear in the repository as theorems in their own right rather than as hypotheses of the main theorems; several are ported, with attribution, from the *PrimeNumberTheoremAnd* project [PNT+]. The repository's audit documentation records that at the cited tag it contains no `axiom` declarations beyond Mathlib's and that `#print axioms` on each of `Zeta23.two_thirds_on_critical_line`, `Zeta23.thmB₀_mult`, `Zeta23.thmC₀_mult`, and the corresponding declarations for the Montgomery–Taylor constants and for Theorem B, returns only the three standard axioms `propext`, `Classical.choice`, `Quot.sound`, with no `sorry`. The repository is available at `https://github.com/anthropics/zeta-23-lean`.

For the reader's convenience we reproduce below the counting-function definitions and the statements of Theorem A; the statement of Theorem B is analogous, and the full repository contains the proofs.

```
-- Counting functions (Zeta23/Statement.lean), against Mathlib's riemannZeta:
import Mathlib.NumberTheory.LSeries.RiemannZeta
import Mathlib.Analysis.Analytic.Order
namespace Zeta23

def IsNontrivialZero (ρ : ℂ) : Prop := riemannZeta ρ = 0 ∧ 0 < ρ.re ∧ ρ.re < 1
def zeroMult (ρ : ℂ) : ℕ := (analyticOrderAt riemannZeta ρ).toNat
def zerosIn (T₁ T₂ : ℝ) : Set ℂ := {ρ | IsNontrivialZero ρ ∧ T₁ < ρ.im ∧ ρ.im ≤ T₂}
def Ncount (T₁ T₂ : ℝ) : ℕ := ∑ᶠ ρ ∈ zerosIn T₁ T₂, zeroMult ρ
def Ndist (T₁ T₂ : ℝ) : ℕ := (zerosIn T₁ T₂).ncard
def N0 (T₁ T₂ : ℝ) : ℕ := ∑ᶠ ρ ∈ zerosIn T₁ T₂ ∩ {ρ | ρ.re = 1 / 2}, zeroMult ρ
def N0star (T₁ T₂ : ℝ) : ℕ := (zerosIn T₁ T₂ ∩ {ρ | ρ.re = 1 / 2}).ncard
def N0simple (T₁ T₂ : ℝ) : ℕ := (zerosIn T₁ T₂ ∩ {ρ | ρ.re = 1 / 2} ∩ {ρ | zeroMult ρ =
    1}).ncard
def Nsimple (T₁ T₂ : ℝ) : ℕ := (zerosIn T₁ T₂ ∩ {ρ | zeroMult ρ = 1}).ncard

-- Main theorems (Zeta23/Unconditional.lean, Zeta23/FinalMult.lean); the types carry no
    hypotheses.
theorem two_thirds_on_critical_line :
    ∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (2 / 3 - ε) * (Ncount T (2 * T) : ℝ) ≤ N0star T (2 * T)

theorem two_thirds_on_critical_line_cumulative :
    ∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (2 / 3 - ε) * (Ncount 0 T : ℝ) ≤ N0star 0 T

theorem thmB₀_mult :
    ∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (2 / 3 - ε) * (Ncount T (2 * T) : ℝ) ≤ N0simple T (2 * T)

theorem thmB₀_mult_cumulative :
    ∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (2 / 3 - ε) * (Ncount 0 T : ℝ) ≤ N0simple 0 T

theorem thmC₀_mult :
    ∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (5 / 6 - ε) * (Ncount T (2 * T) : ℝ) ≤ Ndist T (2 * T)

theorem thmC₀_mult_cumulative :
    ∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (5 / 6 - ε) * (Ncount 0 T : ℝ) ≤ Ndist 0 T

end Zeta23

-- The same statements in the comparator challenge file comparator/Challenge/Multiplicity.lean
-- (names two_thirds_simple_on_critical_line, five_sixths_distinct), and, with
-- cMT := √2·tan(1/√2) / (1 + (1/√2)·tan(1/√2)), the constant c₁* of Theorem D:
theorem montgomery_taylor_simple_on_critical_line_mult :
```

```
    ∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (2 - 1 / cMT - ε) * (Ncount T (2 * T) : ℝ) ≤ N0simple T (2
    * T)

theorem montgomery_taylor_distinct_mult :
    ∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (3 / 2 - cMT⁻¹ / 2 - ε) * (Ncount T (2 * T) : ℝ) ≤ Ndist T
    (2 * T)
```

Von Neumann's trace inequality for Hermitian matrices and both directions of Sylvester's law of inertia for Hermitian forms were not previously in Mathlib and are contributed by this formalisation.

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