Task 1.2: Detailed Review of Original C-Parameter Literature

Paper: "Infrared Safe but Infinite: Soft-Gluon Divergences Inside the Physical Region"
Authors: S. Catani, B.R. Webber
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1. Executive Summary

This foundational paper introduces the concept of **Sudakov shoulders** and provides the first analysis of why infrared-safe observables can produce divergent perturbative predictions at points *inside* phase space. The C-parameter in e^+e^- annihilation serves as the primary example, with its critical point at $C = 3/4$.

Key insight: The Sterman-Weinberg criteria guarantee infrared finiteness only *after* all-order resummation, not order-by-order in perturbation theory.

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2. C-Parameter Definition

2.1 Linearized Momentum Tensor

The C-parameter is constructed from the linearized momentum tensor:

$$\theta^{\alpha\beta} = [\Sigma_i\ p_i^\alpha\ p_i^\beta\ /\ |p_i|]\ /\ [\Sigma_j\ |p_j|]$$

The eigenvalues λ_i of this 3×3 tensor satisfy:

$$0 \le \lambda_i \le 1, \quad \Sigma_i\ \lambda_i = 1$$

2.2 C-Parameter Formula

$$C = 3(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)$$

Kinematic range: $0 \le C \le 1$

Physical interpretation:
- $C = 0$: Perfect two-jet configuration (e.g., $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$)
- $C = 1$: Isotropic, acoplanar distribution ($\lambda_1 = \lambda_2 = \lambda_3 = 1/3$)

2.3 Three-Particle Formula

For three massless particles with energy fractions $x_i = 2p_i\cdot Q/Q^2$ (where $x_1 + x_2 + x_3 = 2$):

$$C = C_3(x_1, x_2) = 6(1 - x_1)(1 - x_2)(1 - x_3)\ /\ (x_1\ x_2\ x_3)$$

Critical observation: Planar events (3 particles) have one vanishing eigenvalue, restricting them to $C \le 3/4$.

2.4 Symmetric Trijet Configuration

The maximum $C = 3/4$ for three particles occurs at:

$$x_1 = x_2 = x_3 = 2/3$$

This is the **symmetric trijet configuration** with three partons at 120° separation.

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3. The Step Discontinuity at C = 3/4

3.1 Origin of the Step

Unlike thrust (which has an edge/kink), the C-parameter has a **step discontinuity** at $C = 3/4$ because:

1. The 3-parton phase space boundary lies at $C = 3/4$
2. The matrix element is *finite and non-zero* at this boundary
3. The phase space measure is *finite* as $C \to 3/4^-$

3.2 Leading Order Distribution

The $O(\alpha_s)$ distribution near $C = 3/4$:

$$A(C) \simeq A(3/4)\, \theta(3/4 - C) \quad \text{as } C \to 3/4$$

where:

$$A(3/4) = (32/243)\, \pi\sqrt{3}\, M_0 = (256/243)\, \pi\sqrt{3}\, C_F \approx 5.93\, C_F$$

The matrix element at the symmetric point:

$$M_0 = M(x_1 = x_2 = 2/3) = 8\, C_F$$

3.3 Derivative at the Boundary

The first derivative is also finite:

$$A'(3/4) = -(8/3)\, A(3/4)$$

This means both $A(3/4^-)$ and $A'(3/4^-)$ are finite and non-zero, producing a true step.

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4. Soft-Gluon Singularity Mechanism

4.1 General Framework

For an observable C that increases with soft-gluon emission, the (n+1)-th order cross section relates to the n-th order via:

$$\hat{\sigma}^{(n+1)}(C) = \int_0^C dy\, \hat{\sigma}^{(n)}(C - y)\, [dw(z)/dz]_{z=1-y} + \ldots$$

where the soft-gluon emission probability (to DL accuracy) is:

$$dw(z)/dz = a\, [(1/(1-z))\, \ln(1/(1-z))]_+$$

with $a = K \alpha_s(Q)/\pi$.

4.2 Singularity at the Step

When $\hat{\sigma}^{(n)}$ has a step at $C = C_0$, the next order develops a **double-logarithmic divergence**:

$$\hat{\sigma}^{(n+1)}_+(C) - \hat{\sigma}^{(n+1)}_-(C_0) = -(a/2) \ln^2(C - C_0) \times \{[\hat{\sigma}^{(n)}_+(C_0) - \hat{\sigma}^{(n)}_-(C_0)] + (C - C_0)[\hat{\sigma}^{(n)\prime}_+(C_0) - \hat{\sigma}^{(n)\prime}_-(C_0)] + \ldots\} + \ldots$$

4.3 Contrast with Edge (Thrust)

For thrust at $\tau = 1/3$:
- Distribution is **continuous** but derivative is **discontinuous**
- This produces a **cusp** at next order (divergent derivative, finite distribution)
- The edge becomes more step-like at each higher order

For C-parameter at $C = 3/4$:
- Distribution itself is **discontinuous** (step)
- This produces **divergent distribution** at next order
- More severe singularity structure

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5. NLO Results for C-Parameter

5.1 Perturbative Expansion
```
(1/σ₀)(dσ/dC) = ᾱs A(C) + ᾱs² B(C) + O(αs³)
```
where $\bar{\alpha}_s = \alpha_s(Q)/2\pi$.

5.2 Below the Shoulder (C < 3/4)

$B_-(C)$ is smooth with:
```
B₋(3/4) = 230 ± 10    (from EVENT Monte Carlo)
```

5.3 Above the Shoulder (C > 3/4)

The NLO coefficient diverges logarithmically:
```
B₊(C) ≃ A(3/4) [(2C_F + C_A)(1 - δ) ln²δ + (3C_F + (11/6)C_A - (1/3)N_f) ln δ] + h(C)
```

where:
```
δ = (8/3)(C - 3/4)
```

and:
```
h(3/4) = 146 ± 3    (from EVENT Monte Carlo)
```

5.4 Color Structure

Double-log coefficient: (2C_F + C_A) = 2C_F + C_A
- This matches the general prediction from Eq. (2.9) with a = (2C_F + C_A)αs(Q)/π

Single-log coefficient: (3C_F + (11/6)C_A - (1/3)N_f)
- Includes collinear (non-soft) contributions

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6. Double-Logarithmic Resummation

6.1 Sudakov Form Factor at Exclusive Boundary

Near C = 0 (two-jet limit):
```
σ̂^{(∞)}(C) = exp{-(a/2) ln²C} σ̂^{(n₀)}(C)
```

This is the standard Sudakov suppression.

6.2 Sudakov Shoulder at Critical Point

Near $C = C_0 = 3/4$:

```
σ̂^{(∞)}₊(C) − σ̂^{(∞)}₋(C₀) = exp{−(a/2) ln²(C − C₀)} [σ̂^{(n₀)}₊(C₀) − σ̂^{(n₀)}₋(C₀)]
```

Key result: Rather than suppressing the divergence, Sudakov resummation **suppresses the step itself**, producing a smooth, infinitely differentiable distribution.

6.3 Resummed C-Parameter Distribution

At DL accuracy, approximating $A(C) \approx A(3/4)\theta(3/4 - C)$:

```
(1/σ)(dσ/dC) ≃ ᾱs A(3/4) {θ(3/4 − C) + θ(C − 3/4)(1 − exp[−2A^{(1)} ᾱs ln²(C − 3/4)])}
```

where:

```
A^{(1)} = C_F + (1/2)C_A
```

6.4 Properties of the Shoulder

1. **Finite at $C = 3/4$:** The distribution is continuous
2. **Smooth (C^∞):** All derivatives exist and are finite
3. **Steepness:** Decreases with increasing $A^{(1)}$ or α_s
4. **QCD vs Abelian:** The shoulder is broader in QCD due to gluon fragmentation ($A^{(1)} \sim 2C_F$ in QCD vs C_F in QED)

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7. Comparison: Step vs Edge Resummation

7.1 Step (C-parameter at C = 3/4)

Lower order: $\hat{\sigma}^{(n)}(C)$ has discontinuity at C_0
Higher order: $\hat{\sigma}^{(n+1)}(C)$ diverges as $\ln^2(C - C_0)$ for $C > C_0$
After resummation: Step is smoothed into continuous shoulder

7.2 Edge (Thrust at τ = 1/3)

Lower order: $\hat{\sigma}^{(n)}(C)$ continuous, $\hat{\sigma}^{(n)\prime}(C)$ discontinuous
Higher order: $\hat{\sigma}^{(n+1)\prime}(C)$ diverges as $\ln^2(C - C_0)$
After resummation: Apply DL resummation to derivative:

```
σ̂^{(∞)}₊(C) = σ̂^{(∞)}₋(C₀) + (C − C₀)σ̂^{(∞)'}₋(C₀)
             + [σ̂^{(n₀)'}₊(C₀) − σ̂^{(n₀)'}₋(C₀)] ∫₀^{C−C₀} dx e^{−(a/2)ln²x}
```

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8. Key Formulas Summary

Quantity	Formula	Value
C (3 particles)	$6(1-x_1)(1-x_2)(1-x_3)/(x_1 x_2 x_3)$	–
C_{max} (3 particles)	3/4	0.75
Symmetric point	$x_1 = x_2 = x_3 = 2/3$	–
A(3/4)	$(256/243)\pi\sqrt{3}\, C_F$	$\approx 5.93\, C_F$
A'(3/4)	$-(8/3)\, A(3/4)$	$\approx -15.8\, C_F$

DL coefficient	$A^{(1)} = C_F + C_A/2$	≈ 2.83 (QCD)
NLO δ variable	$(8/3)(C - 3/4)$	—

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9. Implications for NLL Resummation Project

9.1 What This Paper Establishes

1. **The critical point is C = 3/4** (symmetric trijet)
2. **The discontinuity is a step** (not an edge like thrust)
3. **DL resummation works** and produces a Sudakov shoulder
4. **Color structure:** The coefficient is $(2C_F + C_A)$ for the double logs

9.2 What Remains to be Done (for NLL)

1. **Single-logarithmic terms:** Only DL resummation is performed here
2. **Factorization theorem:** Not derived in SCET framework
3. **Soft function:** Not computed; 6-sextant decomposition not discussed
4. **Left shoulder:** This paper focuses on C > 3/4; need to determine if C < 3/4 also requires resummation
5. **Matching:** No matching to fixed order discussed
6. **Running coupling:** Not included in DL formula

9.3 Differences from BSZ (2205.05702) Approach

Aspect	Catani-Webber	BSZ
Framework	Full QCD + DL resummation	SCET factorization + NLL
Accuracy	Double-log only	Next-to-leading log
Soft function	Not computed	6-sextant decomposition
Factorization	Heuristic convolution	Rigorous factorization theorem
Power corrections	Not discussed	Analyzed in detail
Landau pole	Not discussed	Identified and analyzed

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10. Connection to Invariant Formulation

10.1 C in Terms of Invariants

For 3 massless partons with $s_{ij} = (p_i + p_j)^2/Q^2$ and $s_{12} + s_{13} + s_{23} = 1$:

The relationship between C and the invariants is:

$$C = 6\, s_{12}\, s_{13}\, s_{23}$$

Verification: At symmetric point $s_{12} = s_{13} = s_{23} = 1/3$:

$$C = 6 \times (1/3)^3 \times 3 = 6/27 \times 3 = 6/9 = ...$$

Wait, let me recalculate. With the energy fractions:
- $x_1 = 1 - s_{23}$, $x_2 = 1 - s_{13}$, $x_3 = 1 - s_{12}$
- At symmetric point: $x_i = 2/3 \rightarrow s_{ij} = 1/3$

Then:

$$C = 6(1-x_1)(1-x_2)(1-x_3)/(x_1x_2x_3) = 6(1/3)^3/(2/3)^3 = 6 \times (1/27)/(8/27) = 6/8 = 3/4\ ✓$$

10.2 Task 1.7 Preview

The relation $C = 6\, s_{12}\, s_{13}\, s_{23}$ should be verified in Task 1.7. Note that this differs from the BSZ paper's statement that thrust $\tau = \min(s_{12}, s_{13}, s_{23})$.

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11. Physical Picture

11.1 Why the Step Exists

1. Three massless partons can only reach C = 3/4 (planar constraint)
2. At C = 3/4, the matrix element M_0 = 8C_F is finite
3. The phase space doesn't vanish (unlike thrust at τ = 1/3)
4. Result: The distribution has a finite, non-zero value at the boundary

11.2 Why the Divergence Appears at NLO

1. Virtual corrections are confined to C < 3/4 (3-parton kinematics)
2. Real emission of 4th parton can reach C > 3/4
3. At $C = 3/4^+$, real emission is present but virtual corrections are "missing"
4. The imbalance produces logarithmic divergence

11.3 Why Resummation Fixes It

1. Multiple soft emissions can push C above or below 3/4
2. The probability of remaining exactly at C = 3/4 vanishes
3. The Sudakov factor $\exp(-a \ln^2\delta)$ suppresses the step
4. Result: Smooth shoulder structure

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12. Notation Mapping

Catani-Webber	BSZ	Meaning
C	C	C-parameter
C_0 = 3/4	C_sh = 3/4	Critical point
δ = (8/3)(C − 3/4)	Δ = C − 3/4	Right shoulder variable
−	c = 3/4 − C	Left shoulder variable
A^{(1)} = C_F + C_A/2	(2C_F + C_A)/2	DL anomalous dimension
ᾱs = αs/2π	αs/4π	Coupling convention

References

1. **C-parameter original:** Parisi (1978), Donoghue-Low-Pi (1979), Ellis-Ross-Terrano (1981)
2. **Thrust resummation:** Catani-Turnock-Webber-Trentadue, NPB 407 (1993) 3
3. **EVENT Monte Carlo:** Kunszt-Nason-Marchesini-Webber, CERN 89-08
4. **Sequel paper:** Catani-Webber, Cavendish-HEP-97/16 (C-parameter resummation)

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