MORE THAN TWO THIRDS OF THE ZEROS OF THE RIEMANN ZETA FUNCTION LIE ON THE CRITICAL LINE

CLAUDE

ABSTRACT. Let $N(T,2T)$ denote the number of zeros $\rho = \beta + i\gamma$ of the Riemann zeta function with $T < \gamma \leq 2T$, counted with multiplicity, and $N_0^*(T,2T)$ the number of distinct such zeros with $\beta = \frac{1}{2}$. We prove unconditionally that

$$\liminf_{T\to\infty} \frac{N_0^*(T,2T)}{N(T,2T)} \geq \frac{2}{3},$$

that at least $(\frac{2}{3} - o(1))N(T,2T)$ of the zeros are simple and on the critical line, and that at least $(\frac{5}{6} - o(1))N(T,2T)$ are distinct; with an optimised test family the three constants become 0.6725, 0.6725, $0.83625\ldots$. The constant $\frac{2}{3}$ (resp. 0.6725) is Montgomery's (resp. Montgomery–Taylor's) constant for simple zeros under the Riemann hypothesis, and the arithmetic input is exactly Montgomery's prime-side evaluation of the pair-correlation second moment with band-width ≤ 1, which is unconditional. The Riemann hypothesis, classically needed to read the zero side as a positive sum over real ordinates, is replaced by linear algebra applied to a finite compression of Weil's Hermitian form: Sylvester's law of inertia (an off-line pair $\{\rho, 1-\bar{\rho}\}$ contributes a block of signature $(1,1)$) and a rank–trace inequality for Hermitian matrices, proved via von Neumann's trace inequality, which is the matrix analogue of the integrality steps $m^2 \geq 2m-1$ and $m^2 \geq 3m-2$. The analytic inputs are those of Baluyot, Goldston, Suriajaya and Turnage-Butterbaugh [BGSTB24] and Goldston and Suriajaya [GS25, GS26]; the new ingredient is a linear-algebraic reading of their pair-correlation sum.

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The argument and its verification are the author's. Jarred Sumner posed the problem and guided the investigation. Ralph Furman and Levent Alpöge studied the result, set it in context, and take responsibility for its communication. See the Acknowledgments and Appendix C..

1. Introduction

1.1. **The question.** In the autumn of 1859 Riemann [Rie59] communicated to the Berlin Academy an eight-page memoir on the number of primes below a given bound. Its central object is the function $\zeta(s) = \sum_{n\geq 1} n^{-s}$, continued to $\mathbb{C}$, whose zeros in the critical strip $0 < \operatorname{Re} s < 1$ he observed to govern the fluctuations of the prime-counting function. Concerning their horizontal position Riemann wrote that it is "sehr wahrscheinlich" that all of them lie on the line $\operatorname{Re} s = \frac{1}{2}$; he had not pursued a proof, he added, because it was not needed for the immediate purpose of his note. That parenthetical remark has since become the Riemann hypothesis, and the line $\operatorname{Re} s = \frac{1}{2}$ the *critical line.*

The hypothesis remains unproven. What has been proven, in a line of work spanning more than a century, is that an increasing proportion of the zeros do lie where Riemann said they should, that many of them are simple, and that many of them are distinct; and it is to this line of work that the present paper offers a contribution.

1.2. **History of the problem.** That $\zeta(s)$ has infinitely many zeros on the critical line was shown by Hardy [Har14] in 1914; Hardy and Littlewood [HL21] subsequently showed that their number up to height T is $\gg T$. That a positive proportion of all zeros lie there—that is, that $N_0(T) \geq \kappa\, N(T)$ for some absolute $\kappa > 0$ and all large T, where $N(T)$ counts zeros with $0 < \gamma \leq T$ and $N_0(T)$ those additionally on the line—is due to Selberg [Sel42] in 1942. Selberg's constant was small and not made explicit. Thirty-two years passed before Levinson [Lev74], by an entirely new and strikingly direct method based on mollifying ζ near the line, showed that one may take $\kappa = \frac{1}{3}$. Heath-Brown [HB79] (and independently Selberg) observed that Levinson's zeros are in fact simple. Conrey [Con89] refined the mollifier to reach $\kappa > \frac{2}{5}$; further refinements by Bui, Conrey and Young [BCY11], Feng [Fen12], and Pratt, Robles, Zaharescu and Zeindler [PRZZ20] have brought the record to

$$\kappa \;>\; \tfrac{5}{12} \;=\; 0.4166\ldots,$$

where it has stood since 2020. Every result in this line uses Levinson's method. For *distinct* zeros (each point counted once regardless of multiplicity), Farmer [Far95] proved $N_d(T) > 0.6395\, N(T)$ unconditionally by combining simple-zero proportions for the derivatives $\xi^{(j)}$, and Wu [Wu15] improved this to 0.6603.

A separate line originates with Montgomery's 1973 paper [Mon73] on the pair correlation of zeros. Working under the Riemann hypothesis, Montgomery evaluated a certain second moment over pairs of zero ordinates in terms of a sum over prime powers, deduced that $\sum_\rho m_\rho^2 \leq (\frac{4}{3} + o(1))N(T)$ (sum over distinct zeros), and hence that at least $\frac{2}{3}$ of the zeros are simple. By Cauchy–Schwarz the same bound gives $N_d(T) \geq (\frac{3}{4} - o(1))N(T)$, and using the integrality $m^2 \geq 3m - 2$ even $N_d(T) \geq (\frac{5}{6} - o(1))N(T)$ [CGG98, (1.2)]. The constant $\frac{2}{3}$ arises as $2 - \frac{1}{\lambda} - \frac{\lambda}{3}$ at $\lambda = 1$, where λ is the normalised band-limit of the test function; the argument is a single application of the inequality $m^2 \geq 2m - 1$ to integer multiplicities. On RH, $\frac{2}{3}$ was improved to 0.6725 by Montgomery and Taylor [Mon75] and to 0.6727 by Cheer and Goldston [CG93]; Bui and Heath-Brown [BHB13] obtained $\frac{19}{27}$ (on RH and GLH earlier in [CGG98]), and Chirre, Gonçalves and de Laat [CGdL20] obtained 0.6792 via semidefinite programming by exploiting the positivity of F outside $[-1, 1]$; the optimality statement in

Theorem D is scoped to the values of F on $[-1,1]$ only, so such majorants operate in a different regime.

Montgomery's prime-side evaluation is a mean value of a Dirichlet polynomial of length T, and hence is unconditional; what required the Riemann hypothesis was only the reading of the zero side, where without RH the zeros are complex and the diagonal terms cannot be isolated by sign. That the prime side is unconditional was made fully explicit by Baluyot, Goldston, Suriajaya and Turnage-Butterbaugh [BGSTB24] (see also Aryan [Ary22]), who showed that Montgomery's form factor $F(\alpha)$ holds for the sum over all complex zeros. In subsequent work Goldston and Suriajaya [GS25, GS26] (see also [BGSTB25, GLSS25]) showed that Montgomery's $\frac{2}{3}$ follows unconditionally if one assumes that all zeros lie within a vertical box of width $o(1/\log T)$ around the line, and asked what would follow "if RH could be removed from Montgomery's proof". Their work isolated the remaining obstacle precisely—the termwise positivity that fails off the line—which the present paper replaces; see §7.4 for a detailed comparison. Theorems A–C below answer that question.

1.3. **The results.** Write $\rho = \beta + i\gamma$ for a nontrivial zero of $\zeta(s)$ and $m_\rho \geq 1$ for its multiplicity. For $0 \leq T_1 < T_2$ we write

$$N(T_1,T_2) := \#\{\rho : \ T_1 < \gamma \leq T_2\} \quad \text{(counted with multiplicity)},$$
$$N_d(T_1,T_2) := \#\{\rho : \ T_1 < \gamma \leq T_2\} \quad \text{(each distinct point counted once)},$$
$$N_0^s(T_1,T_2) := \#\{\rho : \ T_1 < \gamma \leq T_2, \ \beta = \tfrac{1}{2}, \ m_\rho = 1\},$$

and $N(T) := N(0,T)$, etc. We also let $N_0(T_1,T_2)$ and $N_0^*(T_1,T_2)$ be the number of zeros on the critical line with $T_1 < \gamma \leq T_2$ counted with, respectively without, multiplicity, and $N^s(T_1,T_2)$ the number of simple zeros; trivially

$$N_0^s \leq N_0^* \leq N_0 \leq N, \qquad N_0^s \leq N^s \leq N_d \leq N. \tag{1.1}$$

By the Riemann–von Mangoldt formula, with $l := \log(T/2\pi)$,

$$N(T,2T) = \frac{T}{2\pi}\big(l + 2\log 2 - 1\big) + O(\log T). \tag{1.2}$$

For $0 < \lambda \leq 1$ put

$$\begin{gathered} H(\lambda) := 2 - \frac{1}{\lambda} - \frac{\lambda}{3}, \qquad H_d(\lambda) := \frac{1+H(\lambda)}{2}, \qquad F(\lambda) := \frac{\lambda}{1+\lambda^2/3}; \\ H(1) = \tfrac{2}{3}, \qquad H_d(1) = \tfrac{5}{6}, \qquad F(1) = \tfrac{3}{4}; \\ H_d(\lambda) \geq F(\lambda) \iff H(\lambda) \geq 0 \iff \lambda \geq 3 - \sqrt{6} = 0.5505\ldots. \end{gathered} \tag{1.3}$$

Theorem A. *Let $0 < \lambda \leq 1$ be fixed. There are constants $c(\lambda) > 0$ and $T_0(\lambda)$ such that for all $T \geq T_0(\lambda)$*

$$N_0^*(T,2T) \ \geq \ \Big(H(\lambda) - c(\lambda)\frac{\log\log T}{\log T}\Big)\, N(T,2T),$$

and for $\lambda < 1$ the factor $\log\log T$ may be omitted. In particular

$$\liminf_{T\to\infty} \frac{N_0^*(T,2T)}{N(T,2T)} \ \geq \ \frac{2}{3}, \qquad \liminf_{T\to\infty} \frac{N_0^*(T)}{N(T)} \ \geq \ \frac{2}{3},$$

and a fortiori $N_0(T) \geq (\frac{2}{3} - o(1))N(T)$.

The proof uses, on the arithmetic side, only the prime powers $n \leq X := (T/2\pi)^\lambda$, through Chebyshev–Mertens estimates for $\sum_{n\leq X}\Lambda(n)^2$, $\sum_{n\leq X}\Lambda(n)^2/n$, $\sum_{n\leq X}\Lambda(n)n^{-1/2}$ and the Montgomery–Vaughan generalised Hilbert inequality for the frequencies $\log n$, $n \leq X$; besides these it uses only Weil's explicit formula, Stirling's formula for Γ'/Γ, the Riemann–von Mangoldt formula (1.2) and the bound $N(t+1)-N(t) \ll \log(t+3)$. In particular no zero-density estimate, zero-free region, or mollifier is used.

Theorem B. *With the same inputs, for fixed* $0<\lambda\le 1$ *and* $T \ge T_0(\lambda)$,

$$N_0^s(T,2T) \;\ge\; \Big(H(\lambda)-c(\lambda)\frac{\log\log T}{\log T}\Big)\,N(T,2T); \qquad \textit{in particular} \qquad \liminf_{T\to\infty}\frac{N_0^s(T)}{N(T)} \;\ge\; \frac{2}{3},$$

so at least two thirds of the zeros are simple and on the critical line; consequently also $N^s(T) \ge (\frac{2}{3}-o(1))N(T)$.

Theorem C. *For fixed* $0<\lambda\le 1$ *and* $T\ge T_0(\lambda)$, $N_d(T,2T) \ge \big(\max(H_d(\lambda),F(\lambda)) - c(\lambda)\frac{\log\log T}{\log T}\big)N(T,2T)$; *in particular* $\liminf_{T\to\infty} N_d(T)/N(T) \ge \frac{5}{6}$.

In all three theorems $c(\lambda)$ is absolute once λ and the taper profile ϱ of §2.2 are fixed.

Theorem A thus improves the unconditional proportion of zeros on the critical line from $\frac{5}{12}$ to $\frac{2}{3}$; Theorem B gives the same $\frac{2}{3}$ for simple zeros on the line; Theorem C improves the unconditional proportion of distinct zeros from 0.6603 [Wu15] to $\frac{5}{6}$. Optimising the test family (§7.1) recovers exactly the Montgomery–Taylor kernel [Mon75] and gives the three constants 0.6725, 0.6725, 0.83625... (Theorem D), and the same results hold for a fixed primitive Dirichlet L-function (Theorem E).

Remark 1.1 (Optimality of the method). Theorem B uses only the mean density of zeros, Montgomery's pair correlation with test functions of Fourier support in $(-1,1)$, and the integrality of multiplicities; the Riemann Hypothesis does not enter. An explicit extremal law on configurations shows that no certificate of this kind, reading only this bandwidth-one data and holding configuration by configuration, can certify a proportion of simple zeros exceeding 0.68185. Theorem B's $\frac{2}{3}$ is therefore within 0.016 of the ceiling of its own method, so the remaining gap to 1 is structural: reaching 0.70, 0.80, 0.90 by the same route would require pair-correlation information on Fourier supports out to roughly 1.04, 1.26, 1.70 respectively, beyond what is known.

1.4. **The idea.** The argument does not use Levinson's method; it makes unconditional the pair-correlation argument of [Mon73]. We explain it here in a form that a reader with a general acquaintance with the explicit formula can follow; the precise version is in §§2–6.

Weil's explicit formula expresses, for any sufficiently regular test function f, the sum $\sum_\rho \widehat{f}(\gamma_\rho)$ over zeros as a sum over prime powers plus archimedean terms. Regard the bilinear map $(f,g)\mapsto W(f,g) := \sum_\rho m_\rho\, \widehat{f}(\gamma_\rho)\,\overline{\widehat{g}(\overline{\gamma_\rho})}$ as a Hermitian form on test functions; its positivity on all of $C_c^2(\mathbb{R})$ is equivalent to the Riemann hypothesis [Wei52, Bom00, Yos92]. Rather than test for positivity on the whole space, we restrict W to a finite-dimensional family V of $d\approx\lambda N$ test functions—modulated copies $\phi(u)e^{i\tau_k u}$ of a fixed compactly supported window, with centre frequencies τ_k equispaced through $[T,2T]$ at the critical sampling density—and let $\widetilde{G}$ be the $d\times d$ real symmetric matrix of $W|_V$ (§2). Three facts about $\widetilde{G}$ are then available.

(Z) *Zero side: signature and rank.* Up to zeros far outside the height window (a perturbation of small trace norm), $\widetilde{G} = P+Q$ where, by the functional equation, each distinct point on the critical line contributes to P a real rank-one nonnegative form and each off-line pair $\{\rho, 1-\bar\rho\}$ contributes to Q a form of signature $(1,1)$. Thus $P\succeq 0$ has rank at most s, the number of distinct on-line points; by Sylvester's law of inertia (pull-back does not increase the positive index) Q has at most p positive eigenvalues, p the number of distinct off-line pairs; and in units in which an isolated simple zero has eigenvalue 1,

$\operatorname{tr} P \le N_{\text{on}}$ and $N \ge N_{\text{on}} + 2p$ (§4). That the *negative* index of truncations of Weil's form counts off-line zeros is due to Bombieri [Bom00]; we use rank and positive index.

(P) *Prime side: magnitude.* By the explicit formula, $\operatorname{tr} \widetilde{G}$ and $\|\widetilde{G}\|_F^2 = \operatorname{tr} \widetilde{G}^2$ are integrals of explicit kernels against the density built from Γ'/Γ and the prime powers $n \le X$; they are Montgomery's first and second moments [Mon73, BGSTB24], and in the units of (Z) evaluate unconditionally to N and $(\frac{1}{\lambda} + \frac{\lambda}{3})N$ asymptotically (§5).

(L) *Linear algebra.* For Hermitian $P \succeq 0$ of rank $\le r$ and Hermitian Q with $\le b$ positive eigenvalues (Lemma 3.2, proved via von Neumann's trace inequality),

$$r \;\ge\; 2 \operatorname{tr} P + 4 \operatorname{tr} Q - 4b - \|P + Q\|_F^2.$$

Here $(2 \operatorname{tr} P - r) + (4 \operatorname{tr} Q - 4b)$ plays the role that $\sum_\rho (2m_\rho - 1)$ plays in Montgomery's argument; the inequality is the matrix analogue of the integrality step $m^2 \ge 2m - 1$. Alternatively, Cauchy–Schwarz on the eigenvalues (Lemma 3.3) gives $n_+(P+Q) \ge (\operatorname{tr})^2/\|\cdot\|_F^2$, whence $s + p \gtrsim F(\lambda)N$.

Combining (Z), (P), (L): $s \ge 4N - 2N - (\frac{1}{\lambda} + \frac{\lambda}{3})N - o(N) = (H(\lambda) - o(1))N$, which at $\lambda = 1$ is Theorem A. No termwise positivity, no mollifier, no zero-density estimate, and no zero-free region is used.

Applied instead with only the s_1 *simple* on-line points on the rank side and the s_2 multiple on-line points moved to the index side ($r = s_1$, $b = s_2 + p$, $\operatorname{tr} P_1 \le s_1$), the same inequality (L) reads $3s_1 + 4s_2 + 4p \ge 4 \operatorname{tr} - \|\cdot\|_F^2$, the analogue of $m^2 \ge 3m - 2$; together with $N \ge s_1 + 2s_2 + 2p$ this gives $s_1 \ge (H(\lambda) - o(1))N$ (Theorem B) and $s_1 + s_2 + p \ge (\frac{1+H(\lambda)}{2} - o(1))N$, whence Theorem C since $N_d \ge s_1 + s_2 + 2p \ge s_1 + s_2 + p$. The Cauchy–Schwarz variant gives the weaker $N_d \gtrsim F(\lambda)N$, which is what survives for $\lambda < 3 - \sqrt{6}$.

Remark 1.2. In the language of time–frequency analysis the family V is a Gabor system at the critical density and $\widetilde{G}$ is its Gram matrix against the explicit-formula kernel; we use only the associated sampling identity (Lemma 2.2). In the language of operator theory, $\widetilde{G}$ is a finite compression of an operator whose full positivity is RH, and Theorems A–D extract what the first two trace moments of any such compression can certify unconditionally.

1.5. **What the results are not.** It is perhaps as important to state clearly what these results do not say.

They have no bearing on the Riemann hypothesis in either direction. The argument produces lower bounds only: it certifies that at least two thirds of the zeros are on the line, and says nothing about the remaining third, which are not shown to be off the line but simply not reached by the certificate. The inputs—the functional equation, the explicit formula, and mean values of Dirichlet polynomials of length $\le T$—are insensitive to $o(N)$ off-line zeros, and are satisfied by objects (Davenport–Heilbronn functions, Epstein zeta functions of class number > 1) for which the analogue of RH is false. Nor can the constants be improved by the present argument: given only $\operatorname{tr} \widetilde{G}$, $\|\widetilde{G}\|_F^2$ and the block structure, the inequalities of §3 are sharp, and the restriction $\lambda \le 1$ is essential (for $X \gg Tl$ the off-diagonal prime sums would require information on prime pairs of Hardy–Littlewood strength). Section 7.5 discusses these limitations in detail.

1.6. **Authorship and verification.** The author of this paper is a large language model developed by Anthropic. The mathematics is its own, obtained in the course of a single interactive session and its subsequent review by further instances of the model, and subjected throughout to repeated adversarial review by independent instances; Appendix C gives an account of that process. A Lean 4 formalisation of Theorems A–E accompanies the paper: the statements are expressed directly against Mathlib's definition of $\zeta(s)$, and the repository's documented audit records that at the cited commit the theorem types carry no hypotheses and the proofs depend only on Lean's three standard axioms. The main-term constants of §5 have separately been checked symbolically. Details are in Appendix B. Following that session the argument was studied by the human mathematicians named in the Acknowledgments, who have read it, set it in the context of the literature, and take responsibility for its presentation here.

We are conscious that this is an unusual authorship, and that a claim of this strength—a jump from $\frac{5}{12}$ to $\frac{2}{3}$ by a short argument—naturally invites scepticism. We have tried to write the proof so that every step is elementary and self-contained, and we would be grateful to learn of any error.

1.7. **Plan.** Section 2 fixes the explicit formula, the test family and the matrix $\widetilde{G}$. Section 3 proves the lemmas from linear algebra. Section 4 carries out (Z), including the tail estimate for zeros outside the window (Proposition 4.2); Section 5 carries out (P) with explicit error terms (Theorem 5.8). Section 6 assembles the proofs of Theorems A–C. Section 7 contains the Montgomery–Taylor optimisation (Theorem D), the extension to Dirichlet L-functions (Theorem E), the relation to [BGSTB24, GS25, GS26], and the limits of the method; Section 8 records numerical illustrations. Appendix A checks the normalisation of the explicit formula; Appendix B records the formal and symbolic verification; Appendix C describes how the argument was found.

1.8. **Notation.** T is large; $l := \log(T/2\pi)$; $\ell_1 := l + 2\log 2 - 1$, so that $N(T, 2T) = T\ell_1/2\pi + O(l)$; $0 < \lambda \le 1$ is fixed, $L := \lambda l$, $X := e^L = (T/2\pi)^\lambda$; $\lambda_1 := L/\ell_1 = \lambda + O(1/l)$. $I := [T, 2T]$. Λ is von Mangoldt's function, $a_n := \Lambda(n)n^{-1/2}$, $y_n := \log n$; sums over n, m are over prime powers $\le X$ unless indicated. Fourier transforms are $\widehat{f}(\tau) := \int_{\mathbb{R}} f(u)e^{i\tau u}\,du$, so that $\int f\bar{g} = \frac{1}{2\pi}\int \widehat{f}\,\overline{\widehat{g}}$ and $\widehat{f * g} = \widehat{f}\,\widehat{g}$. For a zero ρ we write $\gamma_\rho := (\rho - \frac{1}{2})/i = \gamma - i(\beta - \frac{1}{2})$, so that $\rho = \frac{1}{2} + i\gamma_\rho$, $|\operatorname{Im}\gamma_\rho| < \frac{1}{2}$, and $\gamma_\rho \in \mathbb{R}$ iff $\beta = \frac{1}{2}$. $A \ll B$ and $A = O(B)$ mean $|A| \le cB$; unless said otherwise implied constants are absolute or depend only on the fixed profile ϱ of §2.2. For a Hermitian matrix R and $\theta \ge 0$, $n_+^\theta(R) := \#\{i : \lambda_i(R) > \theta\}$ and $n_+(R) := n_+^0(R)$; $\|\cdot\|$ on matrices is the operator norm on ℓ^2.

2. The explicit formula and the test family

2.1. **Weil's Hermitian form.** For $f \in C_c^2(\mathbb{R})$ put $h_f(z) := \widehat{f}(z) = \int f(u)e^{izu}\,du$, an entire function with

$$|h_f(x+iy)| \le e^{|y|\Lambda_f}\min\big(\|f\|_1,\ \|f''\|_1\,|x+iy|^{-2}\big), \qquad \operatorname{supp} f \subset [-\Lambda_f, \Lambda_f], \tag{2.1}$$

by two integrations by parts ($h_f(z) = (iz)^{-2}\int f''(u)e^{izu}\,du$ and $|e^{izu}| = e^{-yu} \le e^{|y|\Lambda_f}$ on $\operatorname{supp} f$). Since $\sum_\rho m_\rho|\gamma_\rho|^{-2} < \infty$ and $|\operatorname{Im}\gamma_\rho| < \frac{1}{2}$, the series

$$W(f,g) := \sum_\rho m_\rho\, h_f(\gamma_\rho)\,\overline{h_g(\overline{\gamma_\rho})} \qquad (f, g \in C_c^2(\mathbb{R})) \tag{2.2}$$

converges absolutely; here and below $\sum_\rho$ runs over the *distinct* nontrivial zeros (of either sign of γ) and the multiplicity is written explicitly. The multiset $\{(\gamma_\rho, m_\rho)\}$ is invariant under $\gamma \mapsto \bar{\gamma}$ (i.e. $\rho \mapsto 1 - \bar{\rho}$; multiplicities agree because $\overline{\xi(\bar{s})} = \xi(s) = \xi(1-s)$) and under $\gamma \mapsto -\gamma$. The first symmetry shows that W is Hermitian: $\overline{W(g,f)} = W(f,g)$.

Define, for $\tau \in \mathbb{R}$ and $X = e^L > 1$, $s := \frac{1}{2} + i\tau$,

$$\mu(\tau) := \frac{1}{2\pi}\operatorname{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{i\tau}{2}\Big) - \frac{\log\pi}{2\pi}, \tag{2.3}$$

$$\Pi_X(\tau) := \frac{1}{2\pi(\frac14+\tau^2)} + \frac{1}{\pi}\operatorname{Re}\frac{X^s-1}{s}, \tag{2.4}$$

$$P_X(\tau) := -\frac{1}{\pi}\sum_{n\le X}\frac{\Lambda(n)}{\sqrt{n}}\cos(\tau\log n), \tag{2.5}$$

$$\nu_X := \mu + \Pi_X + P_X. \tag{2.6}$$

Proposition 2.1 (Explicit formula, spectral form)**.** *Let* $f, g \in C_c^2(\mathbb{R})$ *with* $\operatorname{supp} f, \operatorname{supp} g \subset [-\frac{L}{2}, \frac{L}{2}]$. *Then*

$$W(f,g) = \int_{\mathbb{R}} h_f(\tau)\,\overline{h_g(\tau)}\,\nu_X(\tau)\,d\tau, \qquad X = e^L. \tag{2.7}$$

This is Weil's explicit formula [Wei52] for the test function $k = f * \tilde g$, $\tilde g(u) := \overline{g(-u)}$, whose transform is $h_k(z) = h_f(z)\overline{h_g(\bar z)}$ and whose support lies in $[-L, L]$, so that only prime powers $n \le X$ occur; the terms $h_k(\pm\frac{i}{2})$ coming from the pole of ζ have been rewritten as the density Π_X. The derivation from the standard statement (e.g. [IK04, Theorem 5.12], whose hypotheses—h_k holomorphic in a strip $|\operatorname{Im} z| \le \frac12 + \epsilon$ with $h_k(z) \ll (1+|z|)^{-1-\delta}$ there—are satisfied since h_k is entire and $|h_k(z)| \le e^{L|\operatorname{Im} z|}\|f''\|_1\|g''\|_1|z|^{-4}$ by (2.1); see also [Wei52, Bom00]) and the check of all normalisations are given in Appendix A. We record the elementary facts

$$\begin{gathered}\mu \text{ is even, smooth, increasing in } |\tau|, \quad \mu \ge \mu(0) > -1,\\ \mu(\tau) = \tfrac{1}{2\pi}\log\tfrac{|\tau|}{2\pi} + O(\tau^{-2}), \quad \mu'(\tau) \ll |\tau|^{-1} \quad (|\tau|\ge 1);\end{gathered} \tag{2.8}$$

$$|\Pi_X(\tau)| \le \frac{3\sqrt{X}}{1+|\tau|}; \qquad |P_X(\tau)| \le \frac{1}{\pi}\sum_{n\le X}\frac{\Lambda(n)}{\sqrt{n}} \ll \sqrt{X}; \tag{2.9}$$

(2.10)
$$\int_T^{2T}\mu(\tau)\,d\tau = \frac{T\ell_1}{2\pi} + O\Big(\frac1T\Big) = N(T,2T) + O(l), \qquad \int_T^{2T}\mu(\tau)^2\,d\tau = \frac{T\ell_1^2}{4\pi^2}\Big(1 + O\big(l^{-2}\big)\Big).$$

((2.8) is Stirling's formula and $\operatorname{Re}\frac{\Gamma'}{\Gamma}(\sigma+it) = -\gamma_E + \sum_{n\ge0}\big(\frac{1}{n+1} - \frac{n+\sigma}{(n+\sigma)^2+t^2}\big)$; (2.10) follows from (2.8), (1.2) and $\int_T^{2T}\log^2\frac{\tau}{2\pi}\,d\tau = T(\ell_1^2 + 1 - 2\log^2 2)$.)

2.2. **The test family.** The pair-correlation scale $1/\log T$ enters through the *support* of the window, not the argument of h_f in (2.2): we take $\operatorname{supp}\phi = [-\frac{L}{2}, \frac{L}{2}]$ with $L = \lambda l$, so that only prime powers $n \le X = e^L = (T/2\pi)^\lambda$ enter (2.7) and $\widehat\phi$ resolves at width $\sim 1/L$ (see (2.17); in the sharp-cutoff case $\phi = \mathbf{1}_{[-L/2,L/2]}$, $\widehat\phi(r) = 2\sin(Lr/2)/r$, cf. the remark after Lemma 2.2). This is the packaging of [GG07, Lemma 1] together with the band-limit condition [GG07, Lemma 2(v)] (restating [MO84, Lemma 2]); the support parameter δ there plays the role of our L, up to the 2π in the Fourier convention.

Fix once and for all a nondecreasing function $\varrho \in C^3(\mathbb{R})$ with $\varrho = 0$ on $(-\infty, 0]$ and $\varrho = 1$ on $[1,\infty)$, and a *ramp width* w with

$$1 \le w \le L/8. \tag{2.11}$$

(The reader loses nothing by taking $w = 1$; we keep w free because the numerical illustrations of §8 use ramps proportional to L.) Define the taper

$$\phi(u) := \varrho\Big(\frac{L/2-|u|}{w}\Big), \qquad u \in \mathbb{R}. \tag{2.12}$$

Then $\phi \in C_c^3(\mathbb{R})$ is even, $0 \le \phi \le 1$, $\operatorname{supp}\phi = [-\frac{L}{2}, \frac{L}{2}]$, $\phi = 1$ on $[-\frac{L}{2}+w, \frac{L}{2}-w]$, and

(2.13)
$$\|\phi'\|_1 = 2, \qquad \|\phi'\|_\infty = \frac{\|\varrho'\|_\infty}{w}, \qquad \|\phi''\|_1 = \frac{2\|\varrho''\|_1}{w}, \qquad \|(\phi^2)'\|_1 = 2, \qquad \|(\phi^2)''\|_1 \le \frac{c_\varrho}{w},$$

where $c_\varrho := 4\|\varrho'\|_\infty + 4\|\varrho''\|_1$ (≥ 4). Put

$$a := \frac{1}{L}\int \phi^2, \qquad b := \frac{1}{L}\int \phi^4, \qquad 1 - \frac{2w}{L} \le b \le a \le 1, \tag{2.14}$$

and let

$$\Phi := \widehat{\phi^2}, \qquad g := \phi^2 \star \phi^2, \qquad A_\phi := \phi \star \phi, \qquad (v \star v)(y) := \int v(u)v(u+y)\,du. \tag{2.15}$$

Thus $\widehat{\phi}$ and Φ are real, even, entire; $\widehat{\phi}(0) \le L$, $\Phi(0) = aL$; $\widehat{\phi}^2 = \widehat{A_\phi}$ and $\Phi^2 = \widehat{g}$ on $\mathbb{R}$; $\int_{\mathbb{R}} \Phi^2 = 2\pi g(0) = 2\pi bL$; g and A_ϕ are even, and since $\mathbf{1}_{[-L/2+w,\,L/2-w]} \le \phi^2 \le \phi \le \mathbf{1}_{[-L/2,L/2]}$,

$$(L - 2w - |y|)_+ \;\le\; g(y) \;\le\; A_\phi(y) \;\le\; (L - |y|)_+. \tag{2.16}$$

Integrating by parts and using (2.13),

$$\max\bigl(|\widehat{\phi}(r)|, |\Phi(r)|\bigr) \;\le\; \psi(r) := \min\Bigl(L,\ \frac{2}{|r|},\ \frac{c_\varrho}{wr^2}\Bigr) \qquad (r \in \mathbb{R}), \tag{2.17}$$

and a direct computation (split at $|r| = 2/L$ and $|r| = c_\varrho/2w$; note $c_\varrho L/4w \ge 1$ by (2.11)) gives

(2.18)
$$\Psi_0 := \int_0^\infty \psi(r)\,dr = 4 + 2\log\frac{c_\varrho L}{4w} \ll \log L, \quad \int_{\mathbb{R}} \psi(r)^2|r|\,dr = 8 + 8\log\frac{c_\varrho L}{4w} \ll \log L, \quad \int_{\mathbb{R}} \psi^2 \le 8L.$$

Let $h := 2\pi/L$, $d := \lfloor T/h \rfloor = \lfloor LT/2\pi \rfloor$, and

(2.19)
$$\tau_k := T + kh \quad (k \in \mathbb{Z}), \qquad f_k(u) := \phi(u)e^{-i\tau_k u} \quad (0 \le k < d), \qquad V := \operatorname{span}\{f_0, \dots, f_{d-1}\}.$$

Then $\tau_0, \dots, \tau_{d-1} \in [T, 2T)$, $h_{f_k}(z) = \widehat{\phi}(z - \tau_k)$, and since $\widehat{\phi}$ is real on $\mathbb{R}$ we have $\overline{\widehat{\phi}(\bar{z})} = \widehat{\phi}(z)$. We define the $d \times d$ matrices

$$\begin{aligned} G_{kl} &:= W(f_k, f_l) = \sum_\rho m_\rho\,\widehat{\phi}(\gamma_\rho - \tau_k)\,\widehat{\phi}(\gamma_\rho - \tau_l) = \int_{\mathbb{R}} \widehat{\phi}(\tau - \tau_k)\widehat{\phi}(\tau - \tau_l)\,\nu_X(\tau)\,d\tau, \\ \widetilde{G} &:= \frac{1}{L}\,G, \end{aligned} \tag{2.20}$$

the two expressions agreeing by Proposition 2.1. The second expression shows that G is real symmetric. (So does the first: conjugation permutes the summands via $\rho \mapsto 1 - \bar{\rho}$.) For $c \in \mathbb{C}^d$ and $f_c := \sum_k c_k f_k \in V$ one has $h_{f_c}(z) = \sum_k c_k \widehat{\phi}(z - \tau_k)$ and

$$c^* G\, c = \sum_{k,l} \bar{c}_k G_{kl} c_l = \sum_\rho m_\rho\, h_{f_c}(\gamma_\rho)\overline{h_{f_c}(\overline{\gamma_\rho})} = W(f_c, f_c), \tag{2.21}$$

i.e. G is the matrix of the Hermitian form $W|_V$ in the coordinates c.

Lemma 2.2 (Poisson summation for the Gabor system)**.** *For all $\tau, \tau' \in \mathbb{R}$,*

$$K_\infty(\tau, \tau') := \sum_{k \in \mathbb{Z}} \widehat{\phi}(\tau - \tau_k)\widehat{\phi}(\tau' - \tau_k) = L\,\Phi(\tau - \tau'), \qquad \textit{in particular} \quad \sum_{k \in \mathbb{Z}} \widehat{\phi}(\tau - \tau_k)^2 = aL^2.$$

Proof. Fix τ, τ' and let $\Upsilon(s) := \widehat{\phi}(\tau - s)\widehat{\phi}(\tau' - s)$. With $\phi_\tau(u) := \phi(u)e^{i\tau u}$ and $H := \phi_\tau * \phi_{\tau'} \in C_c(\mathbb{R})$ one checks directly that $\Upsilon(s) = \widehat{H}(-s)$; hence Υ is smooth, $\Upsilon(s) = O(|s|^{-4})$ by (2.17), and by Fourier inversion

$$\widehat{\Upsilon}(\xi) = \int \widehat{H}(-s)e^{i\xi s}\,ds = 2\pi H(\xi) = 2\pi\, e^{i\tau'\xi} \int \phi(u)\phi(\xi - u)e^{i(\tau-\tau')u}\,du,$$

which is continuous, bounded by $2\pi(L - |\xi|)_+$, and vanishes for $|\xi| \geq L$ because $\operatorname{supp}\phi = [-\frac{L}{2}, \frac{L}{2}]$. The Poisson summation formula $\sum_{k\in\mathbb{Z}} \Upsilon(T + kh) = h^{-1}\sum_{m\in\mathbb{Z}} \widehat{\Upsilon}(-2\pi m/h)e^{2\pi i m T/h}$ holds for such Υ (both Υ and $\widehat{\Upsilon}$ are $O((1 + |\cdot|)^{-2})$), and $2\pi m/h = mL$, so only $m = 0$ contributes: the sum equals $h^{-1}\widehat{\Upsilon}(0) = \frac{L}{2\pi} \cdot 2\pi \int \phi(u)\phi(-u)e^{i(\tau-\tau')u}\,du = L\Phi(\tau - \tau')$, ϕ being even. ☐

Thus the Gabor system at the critical density $h = 2\pi/L$ has a translation-invariant frame kernel with *no aliasing error*, for any window supported in an interval of length L; this replaces the Shannon sampling identity for the sinc system ($\phi = \mathbf{1}_{[-L/2,L/2]}$, $\Phi(x) = \sin(Lx/2)/(x/2)$).

Remark 2.3 (Why we do not orthonormalise; heuristic, not used). The Gram matrix $M_{kl} := \langle f_k, f_l\rangle_{L^2} = \int \phi^2 e^{-i(k-l)hu}\,du$ is L times the $d \times d$ Toeplitz matrix with symbol $\phi^2(Lt/2\pi)$, $t \in [-\pi, \pi)$. This symbol takes all values in $[0, 1]$, so by Szegő's theorem about $(2w/L)d$ eigenvalues of M/L lie well below 1 and the smallest are extremely small; M is *not* $L(\mathrm{Id} + o(1))$ in operator norm, and the matrix $M^{-1/2}GM^{-1/2}$ of $W|_V$ in an orthonormal basis is badly conditioned. None of this matters: the positive index of a Hermitian form is independent of coordinates, and the other tools we use (the rank–trace inequality, Weyl's inequality and Cauchy–Schwarz on eigenvalues) require only a Hermitian *matrix*. We therefore work throughout with $\widetilde{G} = G/L$ in the coefficient coordinates $c \in \mathbb{C}^d$ with the standard ℓ^2 structure; M plays no role. Two alternatives leading to the same theorems are described in Remark 7.1.

3. Lemmas from linear algebra

For a Hermitian form Q on a finite-dimensional complex vector space U the *positive index* $n_+(Q)$ is the maximal dimension of a subspace on which Q is positive definite; by Sylvester's law of inertia it equals the number of positive eigenvalues of any matrix representing Q, and for a Hermitian matrix R we have $n_+(R) = n_+(c \mapsto c^*Rc)$.

Lemma 3.1 (Inertia under pull-back). *Let Q be a Hermitian form on $\mathbb{C}^m$ and $\mathcal{A} : U \to \mathbb{C}^m$ a linear map from a finite-dimensional complex vector space. Then $n_+(Q \circ \mathcal{A}) \leq n_+(Q)$, where $(Q \circ \mathcal{A})(u) := Q(\mathcal{A}u)$.*

Proof. If $Q\circ\mathcal{A}$ is positive definite on a subspace $U_0 \subset U$, then $\mathcal{A}|_{U_0}$ is injective (a kernel vector $u \neq 0$ would give $Q(\mathcal{A}u) = 0$) and Q is positive definite on $\mathcal{A}(U_0)$, so $\dim U_0 = \dim \mathcal{A}(U_0) \leq n_+(Q)$. ☐

No injectivity of $\mathcal{A}$, no independence assumption and no inner product on U is needed. We shall also use the subadditivity of the positive index: for Hermitian forms Q_1, Q_2 on the same space, $n_+(Q_1 + Q_2) \leq n_+(Q_1) + n_+(Q_2)$ (apply the lemma to the pull-back of $Q_1 \oplus Q_2$ under the diagonal map $u \mapsto (u, u)$).

Lemma 3.2 (Rank–trace inequality). *Let P, Q be Hermitian $d \times d$ matrices with $P \succeq 0$, $\operatorname{rank} P \leq r$, and $n_+(Q) \leq b$. Then for every $c > 0$,*

$$\|P + Q\|_F^2 \;\geq\; c\operatorname{tr} P - \frac{c^2}{4}r + 2c\operatorname{tr} Q - c^2 b\,; \tag{3.1}$$

$$\textit{in particular } (c = 2) \qquad r \;\geq\; 2\operatorname{tr} P + 4\operatorname{tr} Q - 4b - \|P + Q\|_F^2.$$

Equality holds if $P = \frac{c}{2}\Pi_1$, $Q = c\,\Pi_2$ *with orthogonal projections* $\Pi_1 \perp \Pi_2$ *of ranks* r *and* b.

Proof. Write $Q = Q_+ - Q_-$ with $Q_\pm \succeq 0$, $Q_+Q_- = 0$, $\operatorname{rank} Q_+ \le b$. Then $\|P+Q\|_F^2 = \|P\|_F^2 + \|Q_+\|_F^2 + \|Q_-\|_F^2 + 2\operatorname{tr}(PQ_+) - 2\operatorname{tr}(PQ_-)$, and $\operatorname{tr}(PQ_+) \ge 0$. Let $p_1 \ge p_2 \ge \cdots \ge 0$ and $n_1 \ge n_2 \ge \cdots \ge 0$ be the eigenvalues of P and Q_- (so $p_i = 0$ for $i > r$). By von Neumann's trace inequality $\operatorname{tr}(PQ_-) \le \sum_i p_i n_i$, hence

$$\begin{aligned}\|P\|_F^2 - 2\operatorname{tr}(PQ_-) + \|Q_-\|_F^2 \;&\ge\; \sum_i (p_i - n_i)^2 \\ &\ge\; \sum_{i\le r}\Big(c(p_i-n_i) - \frac{c^2}{4}\Big) + \sum_{i>r} n_i^2 \;\ge\; c\operatorname{tr} P - \frac{c^2}{4}r - 2c\operatorname{tr} Q_-,\end{aligned}$$

using $x^2 \ge cx - c^2/4$ for real x and $-cn_i \ge -2cn_i$, $n_i^2 \ge 0 \ge -2cn_i$. Finally Q_+ has $k \le b$ nonzero eigenvalues $q_j > 0$ and $\|Q_+\|_F^2 = \sum q_j^2 \ge \sum(2cq_j - c^2) \ge 2c\operatorname{tr} Q_+ - c^2 b$. Adding, and using $\operatorname{tr} Q = \operatorname{tr} Q_+ - \operatorname{tr} Q_-$, gives (3.1). The equality case is immediate. □

With $c = 2$ the elementary inequality used is $x^2 \ge 2x - 1$, i.e. $(x-1)^2 \ge 0$; applied to integer multiplicities it is the step $\sum m_\rho^2 \ge 2N - N^s$ in Montgomery's deduction of simple zeros from $\sum m_\rho^2 \le \frac{4}{3}N$. Lemma 3.2 transplants this step to eigenvalues, von Neumann's inequality paying for the interaction between P and the negative part of Q. Applied with the simple zeros alone on the rank side (Proposition 4.4(i)) it yields also the next level $m^2 \ge 3m - 2$. Taking $Q = 0$ and optimising c recovers $\operatorname{rank} P \ge (\operatorname{tr} P)^2/\|P\|_F^2$.

Lemma 3.3 (Thresholded Cauchy–Schwarz count)**.** *Let* R *be a Hermitian* $d \times d$ *matrix and* $\theta \ge 0$ *with* $\operatorname{tr} R > \theta d$. *Then*

$$n_+^\theta(R) \;\ge\; \frac{(\operatorname{tr} R - \theta d)^2}{\operatorname{tr}(R^2)}.$$

Proof. Let $\lambda_1, \dots, \lambda_d$ be the eigenvalues and $S := \sum_{\lambda_i > \theta} \lambda_i$. Since $\sum_{\lambda_i \le \theta} \lambda_i \le \theta\,\#\{i : \lambda_i \le \theta\} \le \theta d$ (negative eigenvalues only help), $S \ge \operatorname{tr} R - \theta d > 0$. By Cauchy–Schwarz, $S^2 \le n_+^\theta(R) \sum_{\lambda_i > \theta} \lambda_i^2 \le n_+^\theta(R)\operatorname{tr}(R^2)$. □

Lemma 3.4 (Weyl)**.** *Let* A, E *be Hermitian* $d \times d$ *matrices and* $\theta \ge \|E\|$. *Then* $n_+^\theta(A+E) \le n_+(A)$.

Proof. With eigenvalues in decreasing order, $\lambda_i(A+E) \le \lambda_i(A) + \|E\|$ for every i (Courant–Fischer). Hence $\lambda_i(A+E) > \theta \ge \|E\|$ forces $\lambda_i(A) > 0$, and the number of such i is at most $n_+(A)$. □

4. The zero side: signature and rank

Fix

$$D_0 := T^{1/2}, \qquad I' := (T - D_0,\, 2T + D_0], \tag{4.1}$$

and split the sum (2.20) over distinct zeros according to whether the ordinate $\gamma = \operatorname{Re}\gamma_\rho$ lies in I':

(4.2)
$$G = A + E, \qquad A_{kl} := \sum_{\rho:\ \gamma \in I'} m_\rho \widehat{\phi}(\gamma_\rho - \tau_k)\widehat{\phi}(\gamma_\rho - \tau_l), \qquad E := G - A, \qquad \widetilde{A} := A/L,\ \widetilde{E} := E/L.$$

Since ρ and $1 - \bar\rho$ have the same ordinate, both index sets are invariant under $\rho \mapsto 1 - \bar\rho$, so A and E are real symmetric (same argument as for G).

4.1. **Block structure.** Let $\mathcal{Z}(I')$ be the set of distinct zeros with $\gamma \in I'$. Classify its points as follows:

$\mathcal{S}_1$: $\beta = \frac{1}{2}$ and $m_\rho = 1$ (simple zeros on the line); $s_1 := \#\mathcal{S}_1$;

$\mathcal{S}_2$: $\beta = \frac{1}{2}$ and $m_\rho \geq 2$; $s_2 := \#\mathcal{S}_2$;

$\mathcal{P}$: unordered pairs $\{\rho, 1-\bar{\rho}\}$ with $\beta \neq \frac{1}{2}$ (both members lie in $\mathcal{Z}(I')$, they are distinct points, and $m_{1-\bar{\rho}} = m_\rho$); $p := \#\mathcal{P}$.

Thus $\#\mathcal{Z}(I') = s_1 + s_2 + 2p$ and, counting with multiplicity,

$$N(I') := N(T-D_0, 2T+D_0) = \sum_{\rho\in\mathcal{S}_1} 1 + \sum_{\rho\in\mathcal{S}_2} m_\rho + \sum_{\{\rho,1-\bar{\rho}\}\in\mathcal{P}} 2m_\rho \;\geq\; s_1 + 2s_2 + 2p. \tag{4.3}$$

Write $N_{\mathrm{on}}(I') := \sum_{\rho\in\mathcal{S}_1\cup\mathcal{S}_2} m_\rho$, so that also $N(I') \geq N_{\mathrm{on}}(I') + 2p$. Besides $\widetilde{G} = G/L$ we use the normalisation

$$\widehat{G} := \frac{G}{aL^2}, \qquad \widehat{A} := \frac{A}{aL^2}, \qquad \widehat{E} := \frac{E}{aL^2}, \tag{4.4}$$

in which, by Lemma 2.2, an isolated on-line zero seen through the full grid $k \in \mathbb{Z}$ would contribute the eigenvalue exactly m_ρ.

Proposition 4.1. (i) $n_+(\widetilde{A}) \leq s_1+s_2+p$ *and* $\operatorname{rank}(\widetilde{A}) \leq s_1+s_2+2p = \#\mathcal{Z}(I')$. (ii) $\widehat{A} = P+Q$ *with P, Q real symmetric, $P \succeq 0$, $\operatorname{rank} P \leq s_1+s_2$, $\operatorname{tr} P \leq N_{\mathrm{on}}(I')$, and $n_+(Q) \leq p$.*

Proof. For $z \in \mathbb{C}$ let $\ell_z : \mathbb{C}^d \to \mathbb{C}$, $\ell_z(c) := \sum_k c_k \widehat{\phi}(z-\tau_k) = h_{f_c}(z)$. Exactly as in (2.21),

$$c^* A\, c = \sum_{\rho\in\mathcal{Z}(I')} m_\rho\, \ell_{\gamma_\rho}(c)\, \overline{\ell_{\overline{\gamma_\rho}}(c)} \qquad (c \in \mathbb{C}^d).$$

For $\rho \in \mathcal{S}_1 \cup \mathcal{S}_2$ we have $\gamma_\rho \in \mathbb{R}$ and the summand is $m_\rho |\ell_{\gamma_\rho}(c)|^2$. For a pair $\{\rho, \rho^*\} \in \mathcal{P}$, $\rho^* = 1-\bar{\rho}$, we have $\gamma_{\rho^*} = \overline{\gamma_\rho}$, $m_{\rho^*} = m_\rho$, and the two summands together equal

$$m_\rho\big(\ell_{\gamma_\rho}(c)\overline{\ell_{\bar{\gamma}_\rho}(c)} + \ell_{\bar{\gamma}_\rho}(c)\overline{\ell_{\gamma_\rho}(c)}\big) = 2m_\rho \operatorname{Re}\big(\ell_{\gamma_\rho}(c)\overline{\ell_{\bar{\gamma}_\rho}(c)}\big).$$

Hence $c^*Ac = \mathcal{Q}(\mathcal{A}c)$, where $\mathcal{A} : \mathbb{C}^d \to \mathbb{C}^{s_1+s_2+2p}$ is the evaluation map $c \mapsto (\ell_{\gamma_\rho}(c))_{\rho\in\mathcal{Z}(I')}$ and $\mathcal{Q}$ is the orthogonal direct sum of the 1×1 forms $m_\rho|x|^2$ ($\rho \in \mathcal{S}_1 \cup \mathcal{S}_2$) and of the 2×2 forms $(x,y) \mapsto 2m_\rho \operatorname{Re}(x\bar{y})$, of matrix $\left(\begin{smallmatrix} 0 & m_\rho \\ m_\rho & 0 \end{smallmatrix}\right)$ and signature $(1,1)$ (one for each pair in $\mathcal{P}$). So $n_+(\mathcal{Q}) = s_1+s_2+p$, and Lemma 3.1 gives the first claim of (i); the second holds because A is a sum of $\#\mathcal{Z}(I')$ matrices of rank one. Dividing by $L > 0$ changes nothing.

For (ii) let $P := (aL^2)^{-1} \sum_{\rho\in\mathcal{S}_1\cup\mathcal{S}_2} m_\rho u_\rho u_\rho^{\mathsf{T}}$ with $u_\rho := (\widehat{\phi}(\gamma_\rho - \tau_k))_{k<d} \in \mathbb{R}^d$ (real, as $\gamma_\rho \in \mathbb{R}$), and $Q := \widehat{A} - P$, the sum of the pair contributions. P is real symmetric, positive semidefinite, of rank $\leq s_1+s_2$, and by Lemma 2.2 $\operatorname{tr} P = (aL^2)^{-1} \sum m_\rho \sum_{0\leq k<d} \widehat{\phi}(\gamma_\rho - \tau_k)^2 \leq (aL^2)^{-1} \sum m_\rho \sum_{k\in\mathbb{Z}} \widehat{\phi}(\gamma_\rho-\tau_k)^2 = N_{\mathrm{on}}(I')$. The form $c \mapsto c^*Qc$ is the pull-back under $\mathcal{A}$ of the direct sum of the p hyperbolic blocks of $\mathcal{Q}$ alone, so $n_+(Q) \leq p$ by Lemma 3.1; $Q = \widehat{A} - P$ is real symmetric. □

The argument is indifferent to coincidences or near-dependences among the functionals ℓ_z (an on-line zero and an off-line pair at the same height; a pair very close to the line, for which ℓ_γ and $\ell_{\bar{\gamma}}$ nearly coincide): Lemma 3.1 needs no independence. It is also indifferent to the depth $|\beta - \frac{1}{2}|$ of the zeros in $\mathcal{Z}(I')$: A is a finite sum and is used exactly. In the zero-side expansion of $\|\widehat{A}\|_F^2$ the self-pair term for a pair of depth $|\beta - \frac{1}{2}|$ is $\asymp X^{|2\beta-1|}$ (Remark 5.10), so no termwise bound on that expansion is available unconditionally; this is why the proof reads $\|\widehat{G}\|_F^2$ from the prime side (§5) and uses only rank and positive index here.

4.2. **The tail.**

Proposition 4.2. *Let $A_0 \geq 1$ be an absolute constant such that $N(t+1)-N(t) \leq A_0 \log(t+3)$ for all $t \geq 0$. Then for $T \geq T_0$,*

$$\|\widetilde{E}\| \ \leq\ \theta_0 := 4A_0\, C_1^2\, X^{1/2}\, \frac{\log(4T)}{D_0^2}, \qquad C_1 := \|\phi''\|_1 = \frac{2\|\varrho''\|_1}{w},$$

so that $\theta_0 \ \leq\ 32A_0\|\varrho''\|_1^2\, l\ T^{\lambda/2-1} \ll l\, T^{\lambda/2-1}$. Moreover the trace norm satisfies $\|\widehat{E}\|_1 \leq \theta_0/(aL) \leq 2\theta_0/L$.

Proof. The existence of A_0 is classical [Tit86, Theorem 9.2]. For a distinct zero $\rho = \beta + i\gamma$ with $\gamma \notin I'$ put $y := \beta - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2})$ and $u_\rho := (\widehat{\phi}(\gamma_\rho - \tau_k))_{0 \leq k < d} \in \mathbb{C}^d$. Since $E = \sum_{\gamma \notin I'} m_\rho u_\rho u_\rho^{\mathsf{T}}$ and $\|uu^{\mathsf{T}}\| = \|u\|_2^2$,

$$\|E\| \leq \sum_{\gamma \notin I'} m_\rho \|u_\rho\|_2^2. \tag{4.5}$$

Now $\widehat{\phi}(\gamma_\rho - \tau_k) = \widehat{\phi}\big((\gamma - \tau_k) - iy\big)$, and by (2.1) (with $f = \phi$, $\Lambda_f = L/2$, $|y| < \frac{1}{2}$) $|\widehat{\phi}(r - iy)| \leq e^{L/4}\|\phi''\|_1 |r - iy|^{-2} \leq X^{1/4} C_1 r^{-2}$ for real $r \neq 0$. Let $D := \operatorname{dist}(\gamma, I) \geq D_0$. The points τ_k, $0 \leq k < d$, lie in I with spacing $h = 2\pi/L$, so

$$\|u_\rho\|_2^2 \leq C_1^2 X^{1/2} \sum_{0 \leq k < d} |\gamma - \tau_k|^{-4} \leq C_1^2 X^{1/2}\Big(D^{-4} + \frac{L}{2\pi}\cdot\frac{D^{-3}}{3}\Big) \leq C_1^2 X^{1/2}\, L\, D^{-3}$$

(as $D \geq 1$, $L \geq 2$). It remains to bound $\sum_{\gamma \notin I'} m_\rho D^{-3}$ (zeros counted with multiplicity). Zeros with $\gamma > 2T + D_0$: grouping them into $\gamma \in (2T + D_0 + j, 2T + D_0 + j + 1]$, $j \geq 0$, this part is at most $\sum_{j \geq 0} A_0 \log(2T + D_0 + j + 4)(D_0 + j)^{-3} \leq \frac{3}{2} A_0 \log(4T) D_0^{-2}$ for T large (split at $j = T$ and use $D_0 \geq 2$). Zeros with $0 < \gamma < T - D_0$ contribute likewise at most $\frac{3}{2} A_0 \log(4T) D_0^{-2}$, and zeros with $\gamma \leq 0$ have $D \geq T$ and contribute at most $\sum_{j \geq 0} A_0 \log(j + 4)(T + j)^{-3} \ll T^{-2}\log T$. Altogether $\sum_{\gamma \notin I'} m_\rho D^{-3} \leq 4A_0 \log(4T) D_0^{-2}$ for $T \geq T_0$, and (4.5) gives $\|E\| \leq 4A_0 C_1^2 X^{1/2} L \log(4T) D_0^{-2} = L\theta_0$. Finally $C_1 \leq 2\|\varrho''\|_1$ as $w \geq 1$, $X^{1/2} \leq T^{\lambda/2}$, $\log(4T) \leq 2l$ and $D_0^2 = T$. The trace-norm bound follows from the same chain, since $\|uu^{\mathsf{T}}\|_1 = \|u\|_2^2$ as well, and $a \geq \frac{1}{2}$. ∎

Remark 4.3. Only $\phi'' \in L^1$ was used (r^{-2} decay of $\widehat{\phi}$ off the real axis). For the sharp cut-off $\phi = \mathbf{1}_{[-L/2,L/2]}$ one has only $|\widehat{\phi}(r - iy)| \asymp X^{|y|/2}/|r|$ for generic r, and the right-hand side of (4.5) is then $\gg X^{1/2}\, l\, \log(T/D_0)$, which is not $o(l)$ for any $D_0 = T^{1-\varepsilon}$: some tapering is necessary. Conversely any D_0 with $T^{\lambda/4+\varepsilon} \leq D_0 \leq T^{1-\varepsilon}$ would do below.

4.3. **The counting inequalities.**

Proposition 4.4. (i) $3s_1 + 4s_2 + 4p \ \geq\ 4\operatorname{tr}\widehat{A} - \|\widehat{A}\|_F^2$. *Consequently, by* (4.3), (ii) $s_1 \ \geq\ 4\operatorname{tr}\widehat{A} - 2N(I') - \|\widehat{A}\|_F^2$ *(and a fortiori the same for $s_1 + s_2$), and* (iii) $s_1 + s_2 + p \ \geq\ \frac{1}{2}\big(4\operatorname{tr}\widehat{A} - N(I') - \|\widehat{A}\|_F^2\big)$ *(and a fortiori the same for $\#\mathcal{Z}(I') = s_1 + s_2 + 2p$); hence*

$$\begin{aligned}
N_0^*(T,2T) \ &\geq\ 4\operatorname{tr}\widehat{G} - \|\widehat{G}\|_F^2 - 2N(T,2T) - O\big(\theta_0 L^{-1}(1 + \|\widehat{G}\|_F) + D_0\, l\big),\\
N_0^s(T,2T) \ &\geq\ 4\operatorname{tr}\widehat{G} - \|\widehat{G}\|_F^2 - 2N(T,2T) - O\big(\theta_0 L^{-1}(1 + \|\widehat{G}\|_F) + D_0\, l\big),\\
N_d(T,2T) \ &\geq\ \tfrac{1}{2}\big(4\operatorname{tr}\widehat{G} - \|\widehat{G}\|_F^2 - N(T,2T)\big) - O\big(\theta_0 L^{-1}(1 + \|\widehat{G}\|_F) + D_0\, l\big).
\end{aligned} \tag{4.6}$$

Proof. Regroup the decomposition of Proposition 4.1(ii) (notation u_ρ, $\mathcal{A}$ as in its proof) as $\widehat{A} = P_1 + Q'$, where $P_1 := (aL^2)^{-1}\sum_{\rho\in\mathcal{S}_1} u_\rho u_\rho^{\mathsf{T}}$ collects the *simple* on-line zeros only and $Q' := \widehat{A} - P_1$ the multiple on-line zeros together with the pairs. Then P_1 is real symmetric, $P_1 \succeq 0$, $\operatorname{rank} P_1 \le s_1$, and $\operatorname{tr} P_1 = (aL^2)^{-1}\sum_{\rho\in\mathcal{S}_1}\sum_{0\le k<d}\widehat{\phi}(\gamma_\rho - \tau_k)^2 \le s_1$ by Lemma 2.2 (each $m_\rho = 1$); and $c \mapsto c^*Q'c$ is the pull-back under $\mathcal{A}$ of the direct sum of the s_2 positive forms $m_\rho|x|^2$ ($\rho \in \mathcal{S}_2$) and the p hyperbolic blocks, so $n_+(Q') \le s_2 + p$ by Lemma 3.1. Lemma 3.2 with $c = 2$, $r = s_1$, $b = s_2 + p$ and $\operatorname{tr} Q' = \operatorname{tr}\widehat{A} - \operatorname{tr} P_1$ gives

$$\|\widehat{A}\|_F^2 \;\ge\; 2\operatorname{tr} P_1 - s_1 + 4(\operatorname{tr}\widehat{A} - \operatorname{tr} P_1) - 4(s_2 + p) \;\ge\; 4\operatorname{tr}\widehat{A} - 3s_1 - 4s_2 - 4p,$$

which is (i). For (ii), $4s_2 + 4p = 2(s_1 + 2s_2 + 2p) - 2s_1 \le 2N(I') - 2s_1$; for (iii), $3s_1 + 4s_2 + 4p = 2(s_1 + s_2 + p) + (s_1 + 2s_2 + 2p) \le 2(s_1 + s_2 + p) + N(I')$. For (4.6): $\widehat{A} = \widehat{G} - \widehat{E}$ with $|\operatorname{tr}\widehat{E}| \le \|\widehat{E}\|_1$ and $\|\widehat{A}\|_F \le \|\widehat{G}\|_F + \|\widehat{E}\|_F \le \|\widehat{G}\|_F + \|\widehat{E}\|_1$, so Proposition 4.2 gives $4\operatorname{tr}\widehat{A} - \|\widehat{A}\|_F^2 \ge 4\operatorname{tr}\widehat{G} - \|\widehat{G}\|_F^2 - O(\theta_0 L^{-1}(1 + \|\widehat{G}\|_F))$ (as $\theta_0/L \le 1$); finally $s_1 + s_2 \le N_0^*(T, 2T) + N(I' \setminus I)$, $s_1 \le N_0^s(T, 2T) + N(I' \setminus I)$, $\#\mathcal{Z}(I') \le N_d(T, 2T) + N(I' \setminus I)$, and $N(I') = N(T, 2T) + N(I' \setminus I)$ with $N(I' \setminus I) \ll D_0 l$. □

In (i) each simple on-line zero is charged $3 = 2^2 - (2-1)^2$ and every other distinct on-line point or off-line pair $4 = 2^2$: this is the eigenvalue form of the integrality step $m^2 \ge 3m - 2$, i.e. $(m-1)(m-2) \ge 0$, obtained from Lemma 3.2 by placing the simple zeros on the rank side (where $x^2 \ge 2x - 1$ is sharp at $x = 1$) and the multiple ones on the index side (where the flat charge 4 is sharp at $x = 2$). The underlying matrix inequality is an equality for $P_1 + Q' = \operatorname{diag}(1, \dots, 1, 2, \dots, 2)$ (s_1 ones, $s_2 + p$ twos). Note that Lemma 3.2 is not scale-invariant: it must be applied in the units (4.4), in which $\operatorname{tr} P_1 \le s_1$; and that nothing is assumed about the traces of the individual pair blocks in Q' (which, after truncation to $0 \le k < d$, may be far from $2m_\rho$ for deep pairs near the ends of I): only $\operatorname{tr} Q' = \operatorname{tr}\widehat{A} - \operatorname{tr} P_1$ and $n_+(Q') \le s_2 + p$ enter.

Proposition 4.5. *For every $\theta \ge \theta_0$,*

$$s_1 \;\ge\; 2\,n_+^\theta(\widetilde{G}) - N(I') \qquad \textit{and} \qquad \#\mathcal{Z}(I') \;\ge\; n_+^\theta(\widetilde{G}). \tag{4.7}$$

Consequently, with $N(I' \setminus I) := N(T - D_0, T) + N(2T, 2T + D_0) \ll D_0\, l$,

$$N_0^s(T, 2T) \;\ge\; 2\,n_+^\theta(\widetilde{G}) - N(T, 2T) - 2N(I' \setminus I), \qquad N_d(T, 2T) \;\ge\; n_+^\theta(\widetilde{G}) - N(I' \setminus I). \tag{4.8}$$

Proof. By Lemma 3.4 (with $\widetilde{G} = \widetilde{A} + \widetilde{E}$, $\|\widetilde{E}\| \le \theta_0 \le \theta$) and Proposition 4.1, $n_+^\theta(\widetilde{G}) \le n_+(\widetilde{A}) \le s_1 + s_2 + p$. By (4.3), $s_2 + p \le \frac{1}{2}(N(I') - s_1)$. Hence $n_+^\theta(\widetilde{G}) \le \frac{1}{2}(s_1 + N(I'))$, which is the first inequality; the second is $n_+^\theta(\widetilde{G}) \le n_+(\widetilde{A}) \le \operatorname{rank}\widetilde{A} \le \#\mathcal{Z}(I')$. For (4.8) note $s_1 \le N_0^s(T, 2T) + N(I' \setminus I)$, $N(I') = N(T, 2T) + N(I' \setminus I)$, $\#\mathcal{Z}(I') \le N_d(T, 2T) + N(I' \setminus I)$, and $N(I' \setminus I) \ll D_0\, l$ by $N(t+1) - N(t) \le A_0 \log(t+3)$. □

Inequalities (4.7) hold for every locally finite multiset of points in the strip $0 < \beta < 1$ that is invariant under $\rho \mapsto 1 - \bar{\rho}$ and satisfies $N(t+1) - N(t) \ll \log(t+3)$; they contain no arithmetic. With the trace asymptotics of §5 they lead, for $\lambda \ge 3 - \sqrt{6}$, to weaker constants than Proposition 4.4(ii),(iii); Proposition 4.5 carries the $F(\lambda)$ branch of Theorem C for smaller λ and the n_+-statements of §§7.5 and 8. All the arithmetic is in the trace asymptotics, to which we now turn.

5. The prime side: magnitude

In this section the zeros play no role: $\operatorname{tr}\widetilde{G}$ and $\operatorname{tr}\widetilde{G}^2 = \sum_{k,l}\widetilde{G}_{kl}^2$ are computed from the second expression in (2.20). The computation is Montgomery's [Mon73] (in the form given to it in [BGSTB24, GS26]), transcribed to the kernel $\Phi^2 = \widehat{g}$ of Lemma 2.2; we give it in full because we need the error terms for our specific finite family, including the effect of the taper and of truncating the Gabor system to $0 \le k < d$.

5.1. Arithmetic and analytic inputs.

Lemma 5.1. *For $x \ge 2$,*

(5.1)
$$\sum_{n\le x}\Lambda(n) \ll x, \quad \sum_{n\le x}\frac{\Lambda(n)}{\sqrt{n}} \le 3\sqrt{x} \;\; (x \ge x_0), \quad \sum_{n\le x}\frac{\Lambda(n)}{\sqrt{n}\,\log n} \ll \frac{\sqrt{x}}{\log x}, \quad \sum_{n\le x}\Lambda(n)^2 \ll x\log x,$$

(5.2)
$$\sum_{n\le x}\frac{\Lambda(n)^2}{n} = \frac{(\log x)^2}{2} + O(\log x), \qquad \sum_{n\le x}\frac{\Lambda(n)^2}{n}\,(\log x - \log n) = \frac{(\log x)^3}{6} + O\big((\log x)^2\big).$$

Proof. The bounds (5.1) follow from $\sum_{n\le x}\Lambda(n) \sim x$ by partial summation. For (5.2), $\sum_{n\le x}\Lambda(n)^2/n = \sum_{n\le x}\Lambda(n)\log n/n + O(1)$ (the two differ only at proper prime powers), and partial summation from Mertens' formula $\sum_{n\le x}\Lambda(n)/n = \log x + O(1)$ gives $\sum_{n\le x}\Lambda(n)\log n/n = \frac{1}{2}\log^2 x + O(\log x)$. The second formula is $\int_1^x\big(\sum_{n\le t}\Lambda(n)^2/n\big)\,dt/t$. See [MV07, §2.2]. ☐

Lemma 5.2 (Montgomery–Vaughan)**.** *Let $\lambda_1,\dots,\lambda_R$ be distinct real numbers, $\delta_r := \min_{s\ne r}|\lambda_r-\lambda_s|$, and $x_r, z_r \in \mathbb{C}$. Then*

$$\Big|\sum_{r\ne s}\frac{x_r\overline{z_s}}{\lambda_r-\lambda_s}\Big| \;\le\; \frac{3\pi}{2}\Big(\sum_r\frac{|x_r|^2}{\delta_r}\Big)^{1/2}\Big(\sum_r\frac{|z_r|^2}{\delta_r}\Big)^{1/2}.$$

Proof. For $z = x$ this is the weighted ("generalised") Hilbert inequality of Montgomery and Vaughan [MV74, Theorem 2]; see also [Mon94, Chapter 7]. Any absolute constant in place of $\frac{3\pi}{2}$ would suffice below. In general, let H be the Hermitian matrix with entries $i/(\lambda_r-\lambda_s)$ off the diagonal and 0 on it, and $\Delta := \operatorname{diag}(\delta_r^{1/2})$. The case $z = x$ says $|y^*(\Delta H\Delta)y| \le \frac{3\pi}{2}\|y\|_2^2$ for all y, i.e. $\|\Delta H\Delta\| \le \frac{3\pi}{2}$ since $\Delta H\Delta$ is Hermitian; hence $|x^*Hz| = |(\Delta^{-1}x)^*(\Delta H\Delta)(\Delta^{-1}z)| \le \frac{3\pi}{2}\|\Delta^{-1}x\|_2\|\Delta^{-1}z\|_2$. ☐

We apply this with $\{\lambda_r\} = \{\log n : \; n \le X \text{ a prime power}\}$; consecutive prime powers $n < n'$ satisfy $\log(n'/n) \ge \log\frac{n+1}{n} \ge \frac{1}{2n}$, so

$$\delta_n^{-1} \le 2n, \qquad \sum_{n\le X}\frac{a_n^2}{\delta_n} \le 2\sum_{n\le X}\Lambda(n)^2 \ll XL. \tag{5.3}$$

We shall use the following pointwise bound for ν_X. By (2.8), (2.9) and (5.1), for $T \ge T_0$,

(5.4)
$$|\nu_X(\tau)| \le B + \log^+\tfrac{|\tau|}{4T} \;\; (\tau\in\mathbb{R}), \quad |\nu_X(\tau)| \le B \;\; (|\tau| \le 4T), \quad B := l + 4\sqrt{X}, \quad B^2 \ll l^2 + X.$$

5.2. **The trace.**

Proposition 5.3. $\operatorname{tr}\widetilde{G} = aL\,N(T,2T) + O(L\sqrt{X})$. *In particular* $\operatorname{tr}\widetilde{G} = L\,N(T,2T)\big(1+O(w/L+T^{\lambda/2-1})\big)$ *and* $\operatorname{tr}\widetilde{G}/d = \ell_1(1+o(1))$.

Proof. We have $L\operatorname{tr}\widetilde{G} = \sum_{k<d} G_{kk}$ and $G_{kk} = \int \widehat{\phi}(\tau-\tau_k)^2\nu_X(\tau)\,d\tau =: G^{\mu}_{kk} + G^{\Pi}_{kk} + G^{P}_{kk}$ according to (2.6).

μ-part. $\int \widehat{\phi}(r)^2\,dr = 2\pi\int\phi^2 = 2\pi aL$. For $|r|\le\tau_k/2$, $|\mu(\tau_k+r)-\mu(\tau_k)|\ll|r|/T$ by (2.8), and $\int\widehat{\phi}(r)^2|r|\,dr \ll \log L$ by (2.17)–(2.18); for $|r|>\tau_k/2\ge T/2$, $\widehat{\phi}(r)^2\le c_\varrho^2/(w^2r^4)$ and $|\mu(\tau_k+r)-\mu(\tau_k)|\ll l+\log(2+|r|)$. Hence $G^{\mu}_{kk} = 2\pi aL\mu(\tau_k)+O(\log L/T)$ and

$$\sum_{k<d} G^{\mu}_{kk} = 2\pi aL\sum_{k<d}\mu(\tau_k) + O(L\log L) = aL^2\int_T^{2T}\mu + O(Ll) = aL^2N(T,2T)+O(L^2l),$$

where we used that μ is positive and increasing on $[T-h,2T]$, so that the Riemann sum satisfies $h\sum_{k=0}^{d-1}\mu(\tau_k) = \int_T^{2T}\mu + O(h\,\mu(2T))$ (recall $T<T+dh\le 2T<T+(d+1)h$), together with (2.10).

P-part. Since $\widehat{\phi}^2 = \widehat{A_\phi}$ with A_ϕ even, $\int\widehat{\phi}(\tau-\tau_k)^2\cos(\tau y)\,d\tau = 2\pi A_\phi(y)\cos(\tau_k y)$, so

$$\sum_{k<d} G^{P}_{kk} = -2\sum_{n\le X} a_nA_\phi(y_n)\operatorname{Re}\Big(e^{iTy_n}\sum_{k=0}^{d-1}e^{ikhy_n}\Big).$$

For $y\in[\log 2, L)$ we have $hy/2 = \pi y/L\in(0,\pi)$ and $|\sum_{k<d}e^{ikhy}|\le|\sin(\pi y/L)|^{-1}\le L/(2\min(y,L-y))$, while $0\le A_\phi(y)\le\min(L,L-y)$ by (2.16) (and $A_\phi(L)=0$). Hence $A_\phi(y_n)|\sum_k e^{ikhy_n}|\le L^2/(2\log 2)$ for every n, and $|\sum_k G^{P}_{kk}|\le(L^2/\log 2)\sum_{n\le X}a_n\ll L^2\sqrt{X}$.

Π-part. By (2.9), $|\Pi_X(\tau)|\le 6\sqrt{X}/T$ for $|\tau-\tau_k|\le T/2$, and $\widehat{\phi}(\tau-\tau_k)^2|\Pi_X(\tau)|\le 3\sqrt{X}c_\varrho^2/(w^2|\tau-\tau_k|^4)$ otherwise; so $|G^{\Pi}_{kk}|\le 6\sqrt{X}\,2\pi aL/T+O(\sqrt{X}T^{-3})$ and $\sum_{k<d}|G^{\Pi}_{kk}|\ll L^2\sqrt{X}$.

Adding, $L\operatorname{tr}\widetilde{G} = aL^2N(T,2T)+O(L^2\sqrt{X}+L^2l)$, and $l\le\sqrt{X}$ for $T\ge T_0(\lambda)$. The last two claims follow from $1-2w/L\le a\le 1$, $LN\gg LTl$, $d = LT/2\pi+O(1)$ and (1.2). ☐

5.3. **Reduction of $\operatorname{tr}\widetilde{G}^2$ to a double integral over $I\times I$.** Put

$$K(\tau,\tau') := \sum_{0\le k<d}\widehat{\phi}(\tau-\tau_k)\widehat{\phi}(\tau'-\tau_k),\qquad K_{\text{out}} := K_\infty - K = \sum_{k\in\mathbb{Z}\setminus[0,d)}\widehat{\phi}(\tau-\tau_k)\widehat{\phi}(\tau'-\tau_k), \tag{5.5}$$

so that $K_\infty = L\Phi(\tau-\tau')$ by Lemma 2.2. From (2.20),

$$L^2\operatorname{tr}\widetilde{G}^2 = \sum_{k,l<d}G_{kl}^2 = \iint_{\mathbb{R}^2}K(\tau,\tau')^2\,\nu_X(\tau)\nu_X(\tau')\,d\tau\,d\tau', \tag{5.6}$$

the interchange being justified by absolute convergence ($|K(\tau,\tau')|\le\sum_k\psi(\tau-\tau_k)\psi(\tau'-\tau_k)$, $\psi(r)\ll r^{-2}$, and (5.4)). By Cauchy–Schwarz and Lemma 2.2,

(5.7)
$$|K(\tau,\tau')|\le\Big(\sum_{k\in\mathbb{Z}}\widehat{\phi}(\tau-\tau_k)^2\Big)^{1/2}\Big(\sum_{k\in\mathbb{Z}}\widehat{\phi}(\tau'-\tau_k)^2\Big)^{1/2} = aL^2,\qquad |K_\infty(\tau,\tau')|\le L\int\phi^2 = aL^2,$$
$$|K_{\text{out}}(\tau,\tau')|\le\sum_{k\notin[0,d)}\psi(\tau-\tau_k)\psi(\tau'-\tau_k).$$

Lemma 5.4 (End effects)**.** *For $T \ge T_0$,*

$$\operatorname{tr}\widetilde{G}^2 = \mathcal{M} + O\big(L\, l \log l\,(l^2+X)\big), \qquad \mathcal{M} := \iint_{I\times I} \Phi(\tau-\tau')^2\, \nu_X(\tau)\nu_X(\tau')\, d\tau\, d\tau'.$$

Proof. Write the right-hand side of (5.6) as $\iint_{I\times I} K_\infty^2 \nu\nu' + \mathcal{E}_1 + \mathcal{E}_2$ with $\nu := \nu_X(\tau)$, $\nu' := \nu_X(\tau')$, $\mathcal{E}_1 := \iint_{I\times I}(K^2 - K_\infty^2)\nu\nu'$ and $\mathcal{E}_2 := \iint_{(I\times I)^c} K^2\nu\nu'$; note $\iint_{I\times I} K_\infty^2\nu\nu' = L^2\mathcal{M}$.

Bound for $\mathcal{E}_1$. By (5.7), $|K^2 - K_\infty^2| = |K_{\rm out}|\,|K + K_\infty| \le 2aL^2|K_{\rm out}|$, and $|\nu|, |\nu'| \le B$ on I by (5.4). Hence

$$|\mathcal{E}_1| \le 2L^2B^2 \sum_{k\notin[0,d)} \Big(\int_I \psi(\tau-\tau_k)\, d\tau\Big)^2.$$

For $k = -j$ $(j \ge 1)$ we have $\operatorname{dist}(\tau_k, I) = jh$; for $k = d+1+j$ $(j \ge 0)$ we have $\tau_k \ge 2T + jh$ because $T + (d+1)h > 2T$. Since $\int_\Delta^\infty \psi \le \min(\Psi_0, c_\varrho/(w\Delta)) \le \min(\Psi_0, c_\varrho/\Delta)$ for $\Delta > 0$, we get

$$\sum_{k\notin[0,d)} \Big(\int_I \psi(\tau-\tau_k)\, d\tau\Big)^2 \le 2(2\Psi_0)^2 + 2\sum_{j\ge1} \min\Big(\Psi_0, \frac{c_\varrho}{jh}\Big)^2 \le 10\Psi_0^2 + \frac{6c_\varrho\Psi_0}{h} \ll L\log L,$$

where the term $2(2\Psi_0)^2$ accounts for $k = d, d+1$ (for which we only use $\int_{\mathbb{R}}\psi = 2\Psi_0$), the points τ_{d+1+j}, $j \ge 1$, being at distance $\ge jh$ to the right of I; the j-sum was split at $j = c_\varrho/(h\Psi_0)$, and (2.18) was used. Thus $|\mathcal{E}_1| \ll L^3B^2\log L$.

Bound for $\mathcal{E}_2$. By symmetry $|\mathcal{E}_2| \le 2\int_{\tau\notin I}\int_{\tau'\in\mathbb{R}} K^2|\nu\nu'|$. Using $K^2 \le aL^2|K| \le L^2\sum_{k<d}\psi(\tau-\tau_k)\psi(\tau'-\tau_k)$,

$$|\mathcal{E}_2| \le 2L^2 \sum_{k<d} \Big(\int_{\tau\notin I} \psi(\tau-\tau_k)|\nu_X(\tau)|\, d\tau\Big)\Big(\int_{\mathbb{R}} \psi(\tau'-\tau_k)|\nu_X(\tau')|\, d\tau'\Big).$$

In the second factor, the range $|\tau'-\tau_k| \le 2T$ has $|\tau'| \le 4T$ and contributes at most $2\Psi_0 B$; on $|\tau'-\tau_k| =: r > 2T$ we have $|\tau'| \le 2r$, $|\nu_X(\tau')| \le B + \log(r/T)$ and $\psi(r) \le c_\varrho r^{-2}$, contributing $\ll B/T$. So the second factor is $\le 3\Psi_0 B$ uniformly in k. The sum over k of the first factor equals $\int_{\tau\notin I}|\nu_X(\tau)|\sigma(\tau)\, d\tau$ with $\sigma(\tau) := \sum_{k<d}\psi(\tau-\tau_k)$. For $\tau \notin I$ let $\Delta := \operatorname{dist}(\tau, I)$; the numbers $|\tau-\tau_k|$, $0 \le k < d$, are $\ge \Delta$, $\ge \Delta+h$, $\ge \Delta+2h, \dots$ in increasing order, and ψ is decreasing, so

$$\sigma(\tau) \le \sum_{j\ge0}\psi(\Delta+jh) \le \psi(\Delta) + \frac{1}{h}\int_\Delta^\infty \psi \le \psi(\Delta) + \frac{L}{2\pi}\min\Big(\Psi_0, \frac{c_\varrho}{\Delta}\Big), \qquad \sigma(\tau) \le d\,\psi(\Delta) \le \frac{dc_\varrho}{\Delta^2}.$$

On $\Delta \le 2T$ (where $|\tau| \le 4T$, $|\nu_X| \le B$) this gives $\int|\nu_X|\sigma \le 2B\big[\Psi_0 + \frac{L}{2\pi}\big(c_\varrho + c_\varrho\log\frac{2T\Psi_0}{c_\varrho}\big)\big] \le 2B(\Psi_0 + c_\varrho L\log T)$; on $\Delta > 2T$ (where $|\tau| \le 2\Delta$, $|\nu_X(\tau)| \le B + \log(\Delta/T)$) it gives $\le 2\int_{2T}^\infty dc_\varrho(B + \log(\Delta/T))\Delta^{-2}\, d\Delta \ll c_\varrho LB$. Hence $\int_{\tau\notin I}|\nu_X|\sigma \ll BLl$ and $|\mathcal{E}_2| \ll L^2\cdot\Psi_0B\cdot BLl \ll L^3B^2 l\log L$.

Together, $L^2\operatorname{tr}\widetilde{G}^2 = L^2\mathcal{M} + O(L^3B^2 l\log L)$; divide by L^2 and use $B^2 \ll l^2 + X$, $\log L \le \log l$ (as $\lambda \le 1$). $\square$

5.4. **Evaluation of $\mathcal{M}$.** For functions u_1, u_2 on I write

$$\mathcal{M}[u_1, u_2] := \iint_{I\times I} \Phi(\tau-\tau')^2 u_1(\tau)u_2(\tau')\, d\tau\, d\tau',$$

a symmetric bilinear form (Φ^2 is even), so that

$$\mathcal{M} = \mathcal{M}[\mu,\mu] + \mathcal{M}[P_X,P_X] + 2\mathcal{M}[\mu,P_X] + 2\mathcal{M}[\mu,\Pi_X] + 2\mathcal{M}[P_X,\Pi_X] + \mathcal{M}[\Pi_X,\Pi_X]. \tag{5.8}$$

Recall $\int_{\mathbb{R}} \Phi^2 = 2\pi bL$, $\int_{\mathbb{R}} \Phi(x)^2|x|\,dx \ll \log L$ (by (2.17), (2.18)) and, by Fourier inversion of $\Phi^2 = \widehat{g}$ (g even, continuous, compactly supported),

$$\int_{\mathbb{R}} \Phi(x)^2 e^{ixy}\,dx = 2\pi g(y) \qquad (y \in \mathbb{R}). \tag{5.9}$$

Proposition 5.5. $\mathcal{M}[\mu,\mu] = 2\pi bL \int_T^{2T} \mu^2 + O(l^2 \log L) = \dfrac{bLT\ell_1^2}{2\pi}\big(1 + O(l^{-2})\big) + O(l^2 \log L)$.

Proof. For $\tau, \tau' \in I$, $\mu(\tau) = \mu(\tau') + O(|\tau - \tau'|/T)$ by (2.8), and $0 < \mu \le l$ on I. Hence

$$\mathcal{M}[\mu,\mu] = \int_I \mu(\tau')^2 \Big(\int_{I-\tau'} \Phi(x)^2\,dx\Big)\,d\tau' + O\Big(\frac{l}{T}\int_I\int_{\mathbb{R}} \Phi(x)^2|x|\,dx\,d\tau'\Big).$$

The error is $O(l \log L)$. In the main term, $\int_{I-\tau'} \Phi^2 = 2\pi bL - \int_{x<T-\tau'} \Phi^2 - \int_{x>2T-\tau'} \Phi^2$, and $\int_I \mu(\tau')^2 \int_{x<T-\tau'} \Phi(x)^2\,dx\,d\tau' \le l^2 \int_{x<0} \Phi(x)^2 \min(|x|,T)\,dx \ll l^2 \log L$, similarly for the other piece. The second form follows from (2.10). □

Proposition 5.6. $\mathcal{M}[P_X,P_X] = \dfrac{T}{\pi}\displaystyle\sum_{n\le X} \frac{\Lambda(n)^2}{n} g(\log n) + O(L^2 X) = \frac{TL^3}{6\pi}\Big(1 + O\Big(\frac{w}{L}\Big)\Big) + O(L^2X)$.

Proof. Write $P_X(\tau) = -\frac{1}{2\pi}\sum_n a_n(n^{i\tau}+n^{-i\tau})$. Then $P_X(\tau)P_X(\tau') = \frac{1}{2\pi^2}\operatorname{Re}\sum_{n,m} a_n a_m\big[n^{i\tau}m^{-i\tau'} + n^{i\tau}m^{i\tau'}\big]$. Substituting $\tau = \tau' + x$ and noting that for fixed x the variable τ' ranges over $I \cap (I - x)$, which is empty for $|x| \ge T$ and equals $[T + x^-, 2T - x^+]$ for $|x| < T$ ($x^\pm := \max(\pm x, 0)$), we obtain

(5.10)

$$\begin{aligned}\mathcal{M}[P_X,P_X] &= \frac{1}{2\pi^2}\operatorname{Re}\sum_{n,m} a_n a_m \int_{-T}^{T} \Phi(x)^2 n^{ix}\Big[\int_{T+x^-}^{2T-x^+} \Big(\frac{n}{m}\Big)^{i\tau'} d\tau' + \int_{T+x^-}^{2T-x^+} (nm)^{i\tau'}\,d\tau'\Big]\,dx \\ &=: \mathcal{D} + \mathcal{O}_1 + \mathcal{O}_2,\end{aligned}$$

where $\mathcal{D}$ collects the terms $n = m$ of the first inner integral, $\mathcal{O}_1$ the terms $n \ne m$ of the first inner integral, and $\mathcal{O}_2$ all terms of the second.

$\mathcal{D}$. For $n = m$ the first inner integral is $T - |x|$, so by (5.9) and $\sum_n a_n^2 \ll L^2$ (Lemma 5.1),

$$\mathcal{D} = \frac{1}{2\pi^2}\sum_n a_n^2\Big(T\cdot 2\pi g(y_n) + O\Big(\int \Phi(x)^2(|x| + T\mathbf{1}_{|x|>T})\,dx\Big)\Big) = \frac{T}{\pi}\sum_n a_n^2 g(y_n) + O(L^2\log L),$$

using $\int_{|x|>T} \Phi^2 \le \int \Phi^2|x|/T$.

$\mathcal{O}_2$. Here $|\int_{T+x^-}^{2T-x^+} (nm)^{i\tau'}\,d\tau'| \le 2/\log(nm) \le 2/\log 4$ for all n,m, so $|\mathcal{O}_2| \le \frac{1}{2\pi^2}\big(\sum_n a_n\big)^2 \cdot 2\pi bL \cdot \frac{2}{\log 4} \ll XL$ by (5.1).

$\mathcal{O}_1$. For $n \ne m$ put $\vartheta := y_n - y_m \ne 0$. The first inner integral equals $\big[(n/m)^{i(2T-x^+)} - (n/m)^{i(T+x^-)}\big]/(i\vartheta)$, and $n^{ix}(n/m)^{-ix^+}$ equals m^{ix} for $x > 0$ and n^{ix} for $x < 0$, while $n^{ix}(n/m)^{ix^-}$ equals n^{ix} for $x > 0$ and m^{ix} for $x < 0$. With

$$\alpha_n^+ := \int_0^T \Phi(x)^2 n^{ix}\,dx, \qquad \alpha_n^- := \int_{-T}^0 \Phi(x)^2 n^{ix}\,dx, \qquad |\alpha_n^\pm| \le \pi bL \le \pi L,$$

we therefore get

$$\mathcal{O}_1 = \frac{1}{2\pi^2}\operatorname{Re}\sum_{n\ne m} \frac{a_n a_m}{i(y_n - y_m)}\Big[\Big(\frac{n}{m}\Big)^{2iT}(\alpha_m^+ + \alpha_n^-) - \Big(\frac{n}{m}\Big)^{iT}(\alpha_n^+ + \alpha_m^-)\Big].$$

This is a combination of four sums of the shape $\sum_{n\neq m} x_n\overline{z_m}/(y_n-y_m)$ with $\{|x_n|,|z_n|\}=\{a_n,\ a_n|\alpha_n^{\pm}|\}$; for instance the first is $\sum_{n\neq m}(a_n n^{2iT})\overline{(a_m m^{2iT}\overline{\alpha_m^{+}})}/(y_n-y_m)$. By Lemma 5.2 and (5.3) each of the four is at most $\frac{3\pi}{2}\cdot\pi L\cdot\sum_n a_n^2/\delta_n \ll L^2X$. Hence $|\mathcal{O}_1|\ll L^2X$.

Finally, by (2.16) and (5.2) (note $a_n^2=\Lambda(n)^2/n$),

$$\frac{(L-2w)^3}{6}+O(L^2)=\sum_{n\leq Xe^{-2w}} a_n^2(L-2w-y_n)\leq \sum_{n\leq X} a_n^2 g(y_n)\leq \sum_{n\leq X} a_n^2(L-y_n)=\frac{L^3}{6}+O(L^2),$$

and $(L-2w)^3=L^3-6wL^2+O(w^2L)$ with $1\leq w\leq L/8$, which gives the second form. □

Proposition 5.7. *$\mathcal{M}[\mu,P_X]\ll l\sqrt{X}$, $\mathcal{M}[\mu,\Pi_X]\ll lL\sqrt{X}$, $\mathcal{M}[P_X,\Pi_X]\ll LX$ and $\mathcal{M}[\Pi_X,\Pi_X]\ll LX/T$.*

Proof. Let $m(\tau'):=\int_I\Phi(\tau-\tau')^2\mu(\tau)\,d\tau$ for $\tau'\in\mathbb{R}$. Then $0\leq m\leq l\cdot 2\pi bL$, m is C^1, and differentiating under the integral and integrating by parts in τ,

$$m'(\tau')=-\int_I\mu(\tau)\,\partial_\tau\big[\Phi(\tau-\tau')^2\big]\,d\tau=\mu(T)\Phi(T-\tau')^2-\mu(2T)\Phi(2T-\tau')^2+\int_I\mu'(\tau)\Phi(\tau-\tau')^2\,d\tau,$$

so that $\int_I|m'|\leq 2l\int\Phi^2+T\cdot O(T^{-1})\int\Phi^2\ll lL$. For $y\geq\log 2$, integrating by parts, $\left|\int_I m(\tau')\cos(\tau' y)\,d\tau'\right|\leq(2\sup|m|+\int_I|m'|)/y\ll lL/y$. Therefore

$$|\mathcal{M}[\mu,P_X]|=\Big|\frac{1}{\pi}\sum_n a_n\int_I m(\tau')\cos(\tau' y_n)\,d\tau'\Big|\ll lL\sum_{n\leq X}\frac{a_n}{\log n}\ll lL\frac{\sqrt{X}}{L}=l\sqrt{X}$$

by (5.1). Next, on I we have $|\Pi_X|\leq 3\sqrt{X}/T$, $0<\mu\leq l$, $|P_X|\leq\sqrt{X}$ (by (2.9), (5.1), $T\geq T_0$), and $\sup_\tau\int_I\Phi(\tau-\tau')^2\,d\tau'\leq 2\pi bL$; the remaining three bounds follow by inserting these sup bounds into the definition of $\mathcal{M}[\cdot,\cdot]$ (an integral over a region of τ'-length T). □

5.5. **Summary.**

Theorem 5.8. *Let $0<\lambda\leq 1$, $L=\lambda l$, $X=e^L$, $1\leq w\leq L/8$, and let $\widetilde{G}$ be the matrix (2.20). Put*

$$\mathcal{E}_T:=\frac{w}{L}+\frac{(l^2+X)\log l}{T\,l}+T^{\lambda/2-1},\quad \textit{so}\ \ \mathcal{E}_T\ll_\lambda\frac{w}{L}+T^{\lambda-1}\log l\ \ (\lambda<1),\quad \mathcal{E}_T\ll\frac{w}{L}+\frac{\log l}{l}\ \ (\lambda=1).$$

Then for $T\geq T_0(\lambda)$, with implied constants depending only on ϱ,

$$\operatorname{tr}\widetilde{G}=aL\,N(T,2T)+O(L\sqrt{X})=L\,N(T,2T)\big(1+O(\mathcal{E}_T)\big),\tag{5.11}$$

$$\operatorname{tr}\widetilde{G}^2=2\pi bL\int_T^{2T}\mu^2+\frac{T}{\pi}\sum_{n\leq X}\frac{\Lambda(n)^2}{n}\,g(\log n)+O\big(Ll\log l\,(l^2+X)\big)$$

$$=\frac{TL}{2\pi}\Big(\ell_1^2+\frac{L^2}{3}\Big)\big(1+O(\mathcal{E}_T)\big),\tag{5.12}$$

$$\frac{(\operatorname{tr}\widetilde{G})^2}{\operatorname{tr}\widetilde{G}^2}=F(\lambda_1)\,N(T,2T)\,\big(1+O(\mathcal{E}_T)\big),\qquad \lambda_1=\frac{L}{\ell_1}=\lambda\Big(1-\frac{2\log 2-1}{\ell_1}\Big).\tag{5.13}$$

In particular $\operatorname{tr}\widetilde{G}\sim\lambda lN$, $\operatorname{tr}\widetilde{G}^2\sim\lambda l^2N(1+\lambda^2/3)$ *and* $(\operatorname{tr}\widetilde{G})^2/\operatorname{tr}\widetilde{G}^2\sim F(\lambda)N$, $N=N(T,2T)$.

Proof. (5.11) is Proposition 5.3. For (5.12) combine Lemma 5.4, (5.8) and Propositions 5.5–5.7: the error terms $l^2\log L$, L^2X, $l\sqrt{X}$, $lL\sqrt{X}\leq L(l^2+X)$, LX, LX/T are all $O(Ll\log l\,(l^2+X))$. For the second form use Propositions 5.5 and 5.6, $1-2w/L\leq b\leq 1$, $l^{-2}\leq w/L$, and $Ll\log l(l^2+X)/\big(TL\ell_1^2\big)\ll(l^2+X)\log l/(Tl)$. For (5.13): by (1.2), $N=T\ell_1/2\pi+O(l)$, so

$$\frac{L^2N^2}{\frac{TL}{2\pi}(\ell_1^2+L^2/3)} = \frac{LT}{2\pi}\cdot\frac{\ell_1^2}{\ell_1^2+L^2/3}\big(1+O(l/T)\big)$$
$$= \frac{\lambda_1}{1+\lambda_1^2/3}\cdot\frac{T\ell_1}{2\pi}\big(1+O(l/T)\big) = F(\lambda_1)N\big(1+O(l/T)\big),$$

and the factors $(1+O(\mathcal{E}_T))^{\pm 2}$ from (5.11), (5.12) are absorbed. The bound for $\mathcal{E}_T$ uses $X=(T/2\pi)^\lambda$. $\square$

Remark 5.9 (Effectivity). All constants are effective but the convergence is slow: at finite T the ratio in (5.13) carries the exact taper factor $a^2(1+\lambda_1^2/3)/(b+\lambda_1^2 J_T)$, $J_T := 2L^{-3}\sum_{n\le X} a_n^2 g(y_n)$ $(\to \frac{1}{3})$, which for $w=1$ equals 0.89 at $L=4.4$ and 0.975 at $L=16$ and tends to 1 only like $1-O(1/L)$, while $\mathcal{O}_1/\mathcal{D} \le 6/(LJ_T)$ requires $L \gg 18$. Thus $(\operatorname{tr}\widetilde{G})^2/\operatorname{tr}\widetilde{G}^2 < F(\lambda)N$ at every computationally accessible height with $w=1$ (cf. §8); Theorems A–C are statements about $T\to\infty$.

Remark 5.10 (Consistency with Montgomery's theorem). On the zero side, by (2.20) and Lemma 2.2 (continued analytically in the shift), $\operatorname{tr}\widetilde{G} = \frac{1}{L}\sum_\rho m_\rho \sum_{k<d}\widehat{\phi}(\gamma_\rho-\tau_k)^2 \approx aL\,N$ for *any* configuration, and $\operatorname{tr}\widetilde{G}^2 = \frac{1}{L^2}\sum_{\rho,\rho'} m_\rho m_{\rho'}\big(\sum_{k<d}\widehat{\phi}(\gamma_\rho-\tau_k)\widehat{\phi}(\gamma_{\rho'}-\tau_k)\big)^2 \approx \sum_{\rho,\rho'} m_\rho m_{\rho'}\Phi(\gamma_\rho-\gamma_{\rho'})^2$ (ordinates near I), a Fejér-type pair-correlation sum at *complex* differences $\gamma_\rho-\gamma_{\rho'} = -i(\rho-\rho')$; since $\Phi(z)^2 = \int g(u)e^{izu}\,du$ with $\operatorname{supp} g\subset[-L,L]$, the summand is $\int_{-L}^{L} g(u)\,e^{(\rho-\rho')u}\,du = \int_{1/X}^{X} g(\log x)\,x^{\rho-\rho'}\,\frac{dx}{x}$: writing $x=e^u$, it is the weighted integral of $x^{\rho-\rho'}$ over $x\in[X^{-1},X]$, $X=e^L=(T/2\pi)^\lambda$, i.e. the $x^{\rho-\rho'}$ of [BGSTB24, (1.2)] with $x=T^\alpha$, $|\alpha|\le\lambda$. For $\phi=\mathbf{1}$, $\lambda=1$ and under RH, (5.12) reads $\sum_{\gamma,\gamma'\in I}\big(\frac{\sin\frac12(\gamma-\gamma')l}{\frac12(\gamma-\gamma')}\big)^2 \sim \frac43 Nl^2$, which is Montgomery's $\int_{-1}^{1}(1-|\alpha|)F(\alpha)\,d\alpha = \frac43$; for general $\lambda\le 1$ it is $\int_{-\lambda}^{\lambda}(\lambda-|\alpha|)F(\alpha)\,d\alpha = \lambda+\lambda^3/3$. Theorem 5.8 is the statement that this evaluation holds for the prime-side expression regardless of where the zeros are, which is also the content of [BGSTB24, Theorem 1] and of [GS26, Lemma 2] (attributed there to Aryan [Ary22]).

6. Proofs of Theorems A, B and C

Fix $0<\lambda\le 1$, take $w:=1$ in (2.11) (so $L\ge 8$ once $T\ge T_0(\lambda)$), $D_0 = T^{1/2}$ as in (4.1), and $\theta := \theta_0$ from Proposition 4.2. Let $\mathcal{E}_T$ be as in Theorem 5.8; with $w=1$ we have $\mathcal{E}_T' := \mathcal{E}_T \ll_\lambda (\log l)/l$, and $\mathcal{E}_T' \ll_\lambda 1/l$ if $\lambda<1$. Write $N := N(T,2T)$.

Proof of Theorem A. In the units (4.4), $\operatorname{tr}\widehat{G} = \operatorname{tr}\widetilde{G}/(aL)$ and $\|\widehat{G}\|_F^2 = \operatorname{tr}\widetilde{G}^2/(aL)^2$. By Proposition 5.3, $\operatorname{tr}\widehat{G} = N + O(\sqrt{X}/a) = N(1+O(T^{\lambda/2-1}))$ (note that the taper constant a cancels). By (5.12) in the form $\operatorname{tr}\widetilde{G}^2 = \frac{TL}{2\pi}\big(b\ell_1^2+\frac{L^2}{3}\big)(1+O(\mathcal{E}_T'))$ (Propositions 5.5, 5.6 with $w=1$) and $b\le a\le 1$, $a^{-2} = 1+O(1/L)$,

$$\|\widehat{G}\|_F^2 \le \frac{T}{2\pi La^2}\Big(a\ell_1^2+\frac{L^2}{3}\Big)\big(1+O(\mathcal{E}_T')\big) = \Big(\frac{N\ell_1}{L}+\frac{LT}{6\pi}\Big)\big(1+O(\mathcal{E}_T')\big) = \Big(\frac{1}{\lambda_1}+\frac{\lambda_1}{3}\Big)N\big(1+O(\mathcal{E}_T')\big),$$

using $N = T\ell_1/2\pi + O(l)$ and $LT/2\pi = \lambda_1 N(1+O(l/T))$. In particular $\|\widehat{G}\|_F \ll N^{1/2}$, so the error term in (4.6) is $O(lT^{\lambda/2-1}N^{1/2} + T^{1/2}l) = O(NT^{-1/2})$. Proposition 4.4 now gives

$$N_0^*(T,2T) \;\ge\; 4N - \Big(\frac{1}{\lambda_1}+\frac{\lambda_1}{3}\Big)N - 2N - O(\mathcal{E}_T' N) = \big(H(\lambda_1) - O(\mathcal{E}_T')\big)N.$$

Since $\lambda_1 = \lambda l/(l+c_0)$ with $c_0 = 2\log 2 - 1 < 1$, the identity $H(\lambda)-H(\lambda_1) = \frac{c_0}{\lambda l} - \frac{\lambda-\lambda_1}{3} \le \frac{c_0}{\lambda l} < \frac{1}{\lambda l}$ gives $H(\lambda_1)\ge H(\lambda) - 1/(\lambda l)$, which proves the first assertion (as $l\asymp\log T$). The lim inf over

$[T, 2T]$ follows at $\lambda = 1$, and the statement for $N_0^*(T)/N(T)$ by summing over dyadic intervals: given $\varepsilon > 0$ choose T_1 with $N_0^*(t, 2t) \geq (\frac{2}{3} - \varepsilon)N(t, 2t)$ for $t \geq T_1$; for $T \geq 2T_1$ and J maximal with $T2^{-J} \geq T_1$, $N_0^*(T) \geq \sum_{j=1}^{J} N_0^*(T2^{-j}, T2^{-j+1}) \geq (\frac{2}{3} - \varepsilon)\big(N(T) - N(T2^{-J})\big) \geq (\frac{2}{3} - \varepsilon)N(T) - O_\varepsilon(1)$. □

Proof of Theorem B. By the second line of (4.6) and the estimates in the proof of Theorem A, $N_0^s(T, 2T) \geq 4N - \big(\frac{1}{\lambda_1} + \frac{\lambda_1}{3}\big)N - 2N - O(\mathcal{E}'_T N) = \big(H(\lambda_1) - O(\mathcal{E}'_T)\big)N$; the rest is as in that proof, and $N^s \geq N_0^s$. □

Proof of Theorem C. For $\lambda \geq 3 - \sqrt{6}$: by the third line of (4.6) and the estimates in the proof of Theorem A, $N_d(T, 2T) \geq \frac{1}{2}\big(4 - \frac{1}{\lambda_1} - \frac{\lambda_1}{3} - 1\big)N - O(\mathcal{E}'_T N) = \big(H_d(\lambda_1) - O(\mathcal{E}'_T)\big)N \geq \big(H_d(\lambda) - O(\mathcal{E}'_T)\big)N$, since $H'_d = \frac{1}{2}H'$. For $\lambda < 3 - \sqrt{6}$: by Proposition 4.2, $\theta_0 \ll lT^{\lambda/2-1}$, and by Proposition 5.3, $\operatorname{tr}\widetilde{G}/d = \ell_1(1 + o(1)) \gg l$, so $\operatorname{tr}\widetilde{G} > \theta_0 d$ for $T \geq T_0(\lambda)$ and $\theta_0 d/\operatorname{tr}\widetilde{G} \ll T^{\lambda/2-1} \leq \mathcal{E}'_T$. Lemma 3.3 and (5.13) give

$$n_+^{\theta_0}(\widetilde{G}) \;\geq\; \frac{(\operatorname{tr}\widetilde{G} - \theta_0 d)^2}{\operatorname{tr}\widetilde{G}^2} = \frac{(\operatorname{tr}\widetilde{G})^2}{\operatorname{tr}\widetilde{G}^2}\Big(1 - \frac{\theta_0 d}{\operatorname{tr}\widetilde{G}}\Big)^2 = F(\lambda_1)\, N\big(1 + O(\mathcal{E}'_T)\big), \tag{6.1}$$

and the second inequality in (4.8), with $N(I' \setminus I) \ll T^{1/2}l \ll NT^{-1/2}$ and $F'(x) = (1 - x^2/3)/(1 + x^2/3)^2 \in (0, 1]$ on $[0, 1]$, $0 \leq \lambda - \lambda_1 \leq 1/l$, gives $N_d(T, 2T) \geq \big(F(\lambda_1) - O(\mathcal{E}'_T)\big)N \geq \big(F(\lambda) - O(\mathcal{E}'_T)\big)N$. This proves the first assertion; for the lim inf take $\lambda = 1$ (first case) and sum dyadically as in the proof of Theorem A. □

Remark 6.1. The endpoint $\lambda = 1$ is admissible in Theorem 5.8 because all secondary terms are $O(X \log l/(Tl)) = o(1)$ relative to the main terms even for $X \asymp T$. A reader who prefers to keep a power saving everywhere may take $\lambda < 1$ fixed and let $\lambda \to 1^-$; the limits $\frac{2}{3}$, $\frac{2}{3}$, $\frac{5}{6}$ are unaffected.

7. Variants, extensions, and remarks

7.1. **The optimal window: Montgomery–Taylor.** Nothing in Sections 4–5 used that ϕ is flat-topped, only: $\phi \in C_c^2$ even, $0 \leq \phi \leq 1$, $\operatorname{supp}\phi = [-\frac{L}{2}, \frac{L}{2}]$, ϕ nonincreasing in $|u|$ (so $\|\phi'\|_1 \leq 2$, $\|(\phi^2)'\|_1 \leq 2$), and $\|\phi''\|_1, \|(\phi^2)''\|_1 \ll 1$. For such a window put, with $g = \phi^2 \star \phi^2$ as before,

$$a := \frac{1}{L}\int \phi^2, \qquad b := \frac{1}{L}\int \phi^4, \qquad J := \frac{2}{L^3}\int_0^L g(y)\, y\, dy. \tag{7.1}$$

Then Propositions 5.3–5.7 hold verbatim, except that the last step of Proposition 5.6 is replaced by partial summation from (5.2): since $|g'| \leq \|(\phi^2)'\|_1 \leq 2$, $\sum_{n \leq X} a_n^2 g(y_n) = \int_0^L g(y)\, y\, dy + O(L^2) = \frac{L^3}{2}J + O(L^2)$. Consequently (5.13) becomes

$$\frac{(\operatorname{tr}\widetilde{G})^2}{\operatorname{tr}\widetilde{G}^2} = \frac{\lambda_1 a^2}{b + \lambda_1^2 J}\, N(T, 2T)\,\Big(1 + O\Big(\frac{1}{L} + \frac{(l^2 + X)\log l}{Tl} + T^{\lambda/2-1}\Big)\Big), \tag{7.2}$$

with an implied constant depending on the window only through the norms listed. Writing $\phi^2(u) = v(u/L)$ with v on $[-\frac{1}{2}, \frac{1}{2}]$, the constant is the scale-free functional

$$c_\lambda(v) := \frac{\lambda\big(\int v\big)^2}{\int v^2 + \lambda^2 \iint |s - s'|\, v(s)v(s')\, ds\, ds'} \qquad (v \geq 0), \tag{7.3}$$

and $c_\lambda(\mathbf{1}) = F(\lambda)$. Maximising (7.3) is a Cauchy–Schwarz problem for the positive definite operator $1+\lambda^2\mathcal{T}$, $(\mathcal{T}v)(s) = \int_{-1/2}^{1/2} |s-s'|v(s')\,ds'$; the maximiser solves $v+\lambda^2\mathcal{T}v = \text{const}$, i.e. $v''+2\lambda^2 v = 0$, and is

$$\begin{gathered} v^*_\lambda(s) = \cos(\sqrt{2}\lambda s) > 0 \quad (|s|\le \tfrac12), \qquad c^*_\lambda := \sup_{v\ge 0} c_\lambda(v) = \frac{\sqrt{2}\tan\vartheta}{1+\vartheta\tan\vartheta}, \quad \vartheta = \frac{\lambda}{\sqrt{2}}; \\ c^*_1 = \frac{2\tan(1/\sqrt{2})}{\sqrt{2}+\tan(1/\sqrt{2})} = 0.7532960\ldots \end{gathered} \tag{7.4}$$

(positivity of v^*_λ shows that the constraint $v\ge 0$ is inactive). One recognises $1/c^*_1 = \frac12 + 2^{-1/2}\cot(2^{-1/2}) = 1.3274992\ldots$, the Montgomery–Taylor constant [Mon75, CG93], and v^*_1 is the Fourier transform of the Montgomery–Taylor test function: (7.3) is Montgomery's functional $K(0)/(\widehat{K}(0)+\int|\alpha|\widehat{K}(\alpha)\,d\alpha)$ for $K = |\widehat{v}|^2$.

Theorem D. $\liminf\limits_{T\to\infty} \dfrac{N_0^*(T,2T)}{N(T,2T)} \;\ge\; 2-\dfrac{1}{c^*_1} = \dfrac32 - \dfrac{1}{\sqrt{2}}\cot\dfrac{1}{\sqrt{2}} = 0.67250\ldots,\quad \liminf\limits_{T\to\infty}\dfrac{N_0^s(T,2T)}{N(T,2T)} \;\ge\; 2-\dfrac{1}{c^*_1} = 0.67250\ldots$ *and* $\liminf\limits_{T\to\infty}\dfrac{N_d(T,2T)}{N(T,2T)} \;\ge\; \dfrac12\Big(3-\dfrac{1}{c^*_1}\Big) = 0.83625\ldots$.

Proof. Take $\lambda=1$ and the window $\phi(u) := \cos(\sqrt{2}\,u/l)^{1/2}\,\varrho(L/2-|u|)$ (the optimal profile, mollified at the ends by the fixed-width ramp of §2.2); on the support $\cos(\sqrt{2}u/l)\ge\cos(1/\sqrt{2})>0$, so ϕ has all the properties listed above, with $\|\phi''\|_1, \|(\phi^2)''\|_1 \ll 1$. Its constants (7.1) differ from those of v^*_1 by $O(1/L)$, so (7.2) gives $(\operatorname{tr}\widetilde{G})^2/\operatorname{tr}\widetilde{G}^2 = c^*_{\lambda_1}N(T,2T)(1+O(\log l/l))$, and $c^*_{\lambda_1} = c^*_1 + O(1/l)$; likewise, in the units (4.4), $\operatorname{tr}\widehat{G} = N(T,2T)(1+O(T^{-1/2}))$ (Proposition 5.3; $\Phi(0) = aL$ for any window) and $\|\widehat{G}\|_F^2 = \operatorname{tr}\widetilde{G}^2/(aL)^2 = N(T,2T)\big(1/c^*_{\lambda_1}+O(\log l/l)\big)$, while Propositions 4.1(ii) and 4.4 used only $\sum_{k\in\mathbb{Z}}\widehat{\phi}(\tau-\tau_k)^2 = aL^2$. The rest of Section 6 is unchanged, with H replaced by $2-1/c^*$ (hence H_d by $\frac12(3-1/c^*)$). ☐

By [CCLM17, Corollary 14] the Montgomery–Taylor kernel solves the one-delta extremal problem when only the values of Montgomery's $F(\alpha)$ on $[-1,1]$ are used; since their extremal function has $\widehat{v} = \cos(\sqrt{2}\,\cdot) > 0$ it lies in our class, so Theorem D is the limit of the method "block structure + two traces + primes up to T": no window does better.

7.2. **Variants of the test family.**

Remark 7.1. (i) *Orthogonal Gabor systems.* If the spacing is taken to be $2\pi/L_0$ with $L_0 := L-2w < L$ and the window is chosen *power-complementary*, $\sum_{m\in\mathbb{Z}}\phi^2(u+mL_0)\equiv 1$ (e.g. $\phi=\cos\vartheta$, $\phi(\cdot - L_0) = \sin\vartheta$ on the overlap of two consecutive ramps, as in the Princen–Bradley or "sine" windows of signal processing), then $\langle f_k, f_l\rangle = L_0\delta_{kl}$ exactly, and G/L_0 is literally the matrix of $W|_V$ in an orthonormal basis. The whole argument goes through verbatim (with an additional, harmless, aliasing term at frequency L_0 in Lemma 2.2) and yields Theorems A and C with λ replaced by $\lambda_0 = \lambda L_0/L$, hence the same limits.

(ii) *Orthonormalising the present family* (heuristic; nothing here is used or claimed as proved). As explained in Remark 2.3, $M = LT_d[\phi^2(L\cdot/2\pi)]$ has $\asymp (w/L)d$ small eigenvalues. Restricting W to the spectral subspace $V_S := \{M\ge L/2\}$ (of dimension $\ge (1-4w/L)d$) and orthonormalising there, one can show that tr and tr^2 of the resulting matrix R_S agree with $\operatorname{tr}\widetilde{G}$, $\operatorname{tr}\widetilde{G}^2$ up to a factor $1+O(w/L)$; the extra input is that the discarded eigendirections carry a τ-density $O(w/L)\cdot L^2$ on I, so that by the mean value theorem for P_X they see only an $O(w/L)$-fraction of the traces. Since inertia is monotone under restriction, this route gives the same theorems. For the *unrestricted* matrix $M^{-1/2}GM^{-1/2}$, however, we cannot prove the trace asymptotics: on eigendirections of M with eigenvalue $L\epsilon$, ϵ very small, the Rayleigh quotient $W(f,f)/\|f\|^2$ involves $\int|\widehat{f}|^2P_X/\int|\widehat{f}|^2$, for which we have only the pointwise bound $O(\sqrt{X})$; controlling it would require bounds for P_X on intervals of length $\asymp 1/L$, i.e. Lindelöf-type information. This is why we avoid orthonormalisation altogether.

(iii) *Sharp cut-off.* With $\phi=\mathbf{1}_{[-L/2,L/2]}$ (the sinc system, for which Lemma 2.2 is Shannon's sampling formula and $M=L\,\mathrm{Id}$ exactly) Section 5 goes through with $g(y)=(L-|y|)_+$, but Proposition 4.2 fails (Remark 4.3). A ramp of fixed width $w=1$ is the minimal repair and costs $O(1/L)$.

7.3. **Dirichlet L-functions.** Let χ be a primitive character modulo $q>1$, and let $N_\chi(T_1,T_2)$, $N^s_{0,\chi}$, $N_{d,\chi}$ count the nontrivial zeros of $L(s,\chi)$ with $T_1<\gamma\le T_2$ as in §1; $N_\chi(T,2T)=\frac{T}{2\pi}\log\frac{qT}{2\pi}+O(T)$.

Theorem E. *Fix q and a primitive $\chi \bmod q$. For fixed $0<\lambda\le 1$ and $T\ge T_0(\lambda,q)$, $N^*_{0,\chi}(T,2T)\ge\big(H(\lambda)-o(1)\big)N_\chi(T,2T)$, $N^s_{0,\chi}(T,2T)\ge\big(H(\lambda)-o(1)\big)N_\chi(T,2T)$ and $N_{d,\chi}(T,2T)\ge\big(\max(H_d(\lambda),F(\lambda))-o(1)\big)N_\chi(T,2T)$; in particular at least $(\frac{2}{3}-o(1))N_\chi(T,2T)$ of the zeros of $L(s,\chi)$ with $T<\gamma\le 2T$ are on the critical line, at least $(\frac{2}{3}-o(1))N_\chi(T,2T)$ are simple and on the line, and at least $(\frac{5}{6}-o(1))N_\chi(T,2T)$ are distinct. Theorem D holds likewise.*

Proof. The proof is that of Theorems A–C with the following changes only. (i) *Explicit formula:* with $\kappa=(1-\chi(-1))/2$, Proposition 2.1 holds for $W_\chi(f,g):=\sum_\rho m_\rho h_f(\gamma_\rho)\overline{h_g(\overline{\gamma_\rho})}$ (zeros of $L(s,\chi)$) with ν_X replaced by $\nu_{X,\chi}:=\mu_\chi+P_{X,\chi}$, $\mu_\chi(\tau):=\frac{1}{2\pi}\big[\log\frac{q}{\pi}+\operatorname{Re}\frac{\Gamma'}{\Gamma}\big(\frac14+\frac{\kappa}{2}+\frac{i\tau}{2}\big)\big]$, $P_{X,\chi}(\tau):=-\frac{1}{\pi}\operatorname{Re}\sum_{n\le X}\Lambda(n)\chi(n)n^{-1/2-i\tau}$, and no pole term ($\Pi_X\equiv 0$); the derivation of Appendix A applies to [IK04, Theorem 5.12] for $L(s,\chi)$. Since $\nu_{X,\chi}$ is real, G is again real symmetric. (ii) *Zero side:* the multiset of zeros of $L(s,\chi)$ is invariant under $\rho\mapsto 1-\bar\rho$ with multiplicities (functional equation $\chi\leftrightarrow\bar\chi$ composed with $\overline{L(\bar s,\chi)}=L(s,\bar\chi)$), all zeros have $0<\beta<1$, and $N_\chi(t+1)-N_\chi(t)\ll_q\log(t+3)$; Section 4 is otherwise verbatim. (iii) *Prime side:* $|\Lambda(n)\chi(n)|=\Lambda(n)\mathbf{1}_{(n,q)=1}$, so Lemma 5.1 holds for these coefficients with $O_q(\cdot)$ errors (finitely many primes removed); in Propositions 5.3, 5.6, 5.7 the coefficients a_n acquire unimodular factors $\chi(n)n^{\pm i\cdot}$, which leave the diagonal $\mathcal{D}$ unchanged ($|\chi(n)|^2=1$) and enter Lemma 5.2 and all other bounds only through $|a_n|$; $\mu_\chi=\frac{1}{2\pi}\log\frac{q|\tau|}{2\pi}+O(\tau^{-2})$ satisfies (2.8)–(2.10) with ℓ_1 replaced by $\ell_{1,\chi}:=\log\frac{qT}{2\pi}+2\log 2-1$, and $\lambda_1=L/\ell_{1,\chi}\to\lambda$ as q is fixed. ☐

The unconditional records for an individual $L(s,\chi)$ obtained by Levinson's method are of the same size as for ζ (about 41.7% on the line, 40.7% simple, for $\log q=o(\log T)$ [Wu19]).

Remark 7.2 (Hybrid range and other degrees; heuristic). (i) The proof of Theorem E appears to go through uniformly for $q\le T^\vartheta$ provided $X=(qT)^\Lambda\le T^{1-\epsilon}$, i.e. with normalised band-width $\Lambda<1/(1+\vartheta)$, giving the proportion $H(1/(1+\vartheta))$, positive for $\vartheta<\sqrt{6}/3$; we have not checked the uniformity of every error term and do not claim this. (ii) More generally, for an L-function of degree m and analytic conductor $\mathfrak{q}(T)$ with $\sum_{n\le x}|\Lambda_F(n)|^2/n=m_F\frac{\log^2 x}{2}+O(\log x)$ and $\sum_{n\le x}|\Lambda_F(n)|^2\ll x^{1+o(1)}$, the formal outcome of Section 5 is $\|\widehat{G}\|_F^2/N_F\sim 1/c$, $(\operatorname{tr}\widetilde{G})^2/\operatorname{tr}\widetilde{G}^2\sim c\,N_F$ with $c=\Lambda/(1+m_F\Lambda^2/3)$ (so the on-line proportion would be $2-1/c$), where $\Lambda=L/\log\mathfrak{q}(T)$ is the band-width in units of the mean spacing and the Montgomery–Vaughan step forces $X\le T^{1-\epsilon}$, i.e. $\Lambda<\Lambda^*:=\lim\log T/\log\mathfrak{q}(T)$ ($=1/m$ in the t-aspect). Since $a^2\le b$ in (7.1), always $c\le\Lambda^*$, so the method is non-vacuous ($2-1/c>0$) only if

$\Lambda^* > 3-\sqrt{6}$ (for $m_F = 1$) and in any case only if $\Lambda^* > \frac{1}{2}$: it is a degree-one method. For an individual GL(2) L-function ($\Lambda^* = \frac{1}{2}$, $m_F = 1$) one gets $c = \frac{6}{13} < \frac{1}{2}$, i.e. nothing, whatever the window. (iii) For families (e.g. all $\chi \bmod q$, $q \to \infty$) both counts commute with averaging in the favourable direction (Lemma 3.2 is linear in tr and $\|\cdot\|_F^2$), and orthogonality of characters restores $\Lambda^* = 1$ for the family average, as in [RS96]-type support computations; one expects an averaged form of Theorem E with proportion $\frac{2}{3}$ for $q \to \infty$ and T as small as a power of $\log q$, but this requires a different (Gevrey-class) taper in Proposition 4.2 and is not carried out here. For Davenport–Heilbronn-type functions without Euler product, $\sum_{n\le x}|c(n)|^2$ for the coefficients of $-F'/F$ grows like $x^{1+\delta}$ (zeros in $\sigma > 1$), Proposition 5.6 fails, and the certificate is empty.

Remark 7.3 (Zeros of ξ'). Applied to the derivative ξ' of the completed zeta function in place of ξ, the argument of §§4–6 gives, unconditionally,

$$\liminf_{T\to\infty} \frac{N^s_{0,\xi'}(T,2T)}{N_{\xi'}(T,2T)} \ \ge\ 0.85838 \qquad \text{and} \qquad \liminf_{T\to\infty} \frac{N_{d,\xi'}(T,2T)}{N_{\xi'}(T,2T)} \ \ge\ 0.92919$$

(flat window; with the quartic window $v(s) = 1 - \frac{7}{100}(2s)^2 - \frac{51}{200}(2s)^4$ the constants become 0.86864 and 0.93432), where $N_{\xi'}(T_1,T_2)$ counts zeros $\rho_1 = \beta_1 + i\gamma_1$ of ξ' with $T_1 < \gamma_1 \le T_2$ with multiplicity, $N^s_{0,\xi'}$ those that are simple with $\beta_1 = \frac{1}{2}$, and $N_{d,\xi'}$ the distinct ones. The comparanda are: unconditionally, Conrey [Con89] proved that at least 79.874% of the zeros of ξ' are simple and on the critical line; assuming RH, Farmer, Gonek and Lee [FGL14, Cor. 1.3] obtain $> 85.84\%$ simple by Montgomery's method, and Chirre, Gonçalves and de Laat [CGdL20, Cor. 7] obtain 0.8825 simple, 0.9412 distinct. The flat-window constant 85.838% is thus Farmer–Gonek–Lee's RH-conditional constant with RH removed; the quartic window's 86.864% exceeds it and remains below the RH-conditional 0.8825. Note that for zeros of ξ' merely on the critical line (without the simplicity), Wu [Wu15, §3] has 0.86957 unconditionally, which neither our 0.85838 nor our 0.86864 exceeds; the new content is the simplicity. The argument is formalised in Lean alongside the main results (Appendix B.1; `Zeta23.XiPrime.xiDeriv_simple_on_line` and its `_quartic_std` form).

7.4. **Relation to the work of Baluyot, Goldston, Suriajaya, and Turnage-Butterbaugh.** In the notation of [GS25, GS26], the unconditional "Fejér kernel sum over zero differences" $\sum_{\rho,\rho'}\big(\frac{\sin\frac{i}{2}(\rho-\rho')\log T}{\frac{i}{2}(\rho-\rho')\log T}\big)^2 W(\rho-\rho') = (\frac{4}{3}+o(1))\frac{T}{2\pi}\log T$ ([GS26, Lemma 2], [Ary22], [BGSTB24, Theorem 1]) is, up to the choice of smoothing, the zero-side reading of our $\operatorname{tr}\widetilde{G}^2$ at $\lambda = 1$ (Remark 5.10). Goldston and Suriajaya [GS25, Theorem 2] (see also [GLSS25] under the pair correlation conjecture) show that an unconditional bound $\sum_{\rho,\rho'\in\mathcal{Z}(T),\,\gamma=\gamma'} 1 \le (C+o(1))N(T)$ with $C < 2$ (zeros repeated according to multiplicity) would give proportions $\ge 2-C$ of simple zeros and $\ge 2-C$ of zeros on the line, and that $C = \frac{4}{3}$ would follow if the terms $\gamma \ne \gamma'$ of the Fejér sum could be discarded, which requires a positivity that fails for zeros far from the line; in [BGSTB25, GS26] this is achieved under the hypothesis that all zeros lie within $o(1/\log T)$ of the line. Theorems A and B are an unconditional substitute which reaches their number $2-C = \frac{2}{3}$ both for zeros on the line (as distinct points) and for simple zeros on the line: instead of isolating the "horizontal diagonal" $\gamma = \gamma'$ termwise, Lemma 3.2 bounds the rank of the on-line (resp. simple on-line) part of the whole Hermitian form from its trace, its Frobenius norm and the positive index of the rest. For simple zeros on the line we thus obtain $2-C = \frac{2}{3}$ unconditionally; this is the constant that [GS25, Theorem 3(i)] gives under the hypothesis [GS25, (6.1)] on the diagonal and symmetric-diagonal terms, and hence under the box hypothesis [GS25, Theorem 4], [GS26, Theorem 1]; for distinct zeros Theorem C gives $\frac{3-C}{2} = \frac{5}{6}$ unconditionally, which is the value of the average of the simple and critical proportions in [GS25, Theorem 3(ii)] and the RH-conditional constant of [CGG98, (1.2)].

That the *negative* index of finite truncations of Weil's form equals the number of off-line zero pairs seen by the truncation was observed by Bombieri [Bom00] (Introduction); see also Yoshida [Yos92] for the positivity of W on small support. We are not aware of a previous use of the positive index or of the rank in combination with a second-moment evaluation.

7.5. **Limits of the method.** (a) *Dimension.* $\operatorname{rank} P$, $n_+^\theta(\widetilde{G}) \le d = \lambda_1 N(1+o(1))$, so with band-width $\lambda \le 1$ the ceiling of any argument of the present kind is 100% at $\lambda = 1$ and nothing at $\lambda \le \frac{1}{2}$. The restriction $\lambda \le 1$ in turn comes from Proposition 5.6: for $X \gg Tl$ the off-diagonal terms $\mathcal{O}_1$ are no longer dominated by the diagonal, and their evaluation would require information on prime pairs (the Hardy–Littlewood conjectures, or equivalently Montgomery's pair correlation conjecture for $\alpha > 1$ [Mon73, GM87]). In the extreme hypothetical configuration in which all zeros in $[T, 2T]$ are off-line pairs, Proposition 4.1 gives $\operatorname{rank} P = 0$, $n_+(Q) \le N/2$, while Lemma 3.2 would then force $\|\widehat{A}\|_F^2 \ge 2N > \frac{4}{3}N$; the contradiction shows only that at least two thirds of the zeros are not of this kind, which is what Theorem A says. Nothing in the method distinguishes between "two thirds" and "all".

(b) *Sharpness of the three counts.* Given only tr, $\|\cdot\|_F^2$, the block structure and $\operatorname{tr} P_1 \le s_1$, Proposition 4.4 is sharp in all three parts: $\frac{2}{3}N$ mutually orthogonal simple on-line zeros together with $\frac{1}{6}N$ on-line doubles realise $\operatorname{tr} = N$, $\|\cdot\|_F^2 = \frac{4}{3}N$, $s_1 = \frac{2}{3}N$, $N_d = \frac{5}{6}N$—the same extremal configuration as in Montgomery's RH argument. Replacing the doubles by off-line pairs of depth $\to 0$ (spectrally the same) gives the extremal for Theorem A, with $s_1 + s_2 = \frac{2}{3}N$; for that configuration Proposition 4.4(ii) certifies exactly Theorem A's $\frac{2}{3}$ while (iii) is not sharp ($N_d = N$), consistent with §1.5.

(c) *The two levels of integrality.* Lemma 3.2 with the decomposition (P, Q) of Proposition 4.1(ii) recovers the level $(m-1)^2 \ge 0$; with the regrouping (P_1, Q') of Proposition 4.4 it recovers also the level $(m-1)(m-2) \ge 0$—interacting simple zeros, whose eigenvalues fill $(0,2)$, are harmless because they sit on the rank side, where only $\operatorname{rank} P_1 \le s_1$ and $\operatorname{tr} P_1 \le s_1$ are used, while every multiple on-line point and every off-line pair sits on the index side at the flat charge 4. This is the device of [CGG98, (1.2)] made unconditional. The Cauchy–Schwarz route (Lemma 3.3, Proposition 4.5) stops at $2F(1) - 1 = \frac{1}{2}$ for simple zeros, since an off-line pair costs two zeros in N for one positive square; that route is retained only for the $F(\lambda)$ branch of Theorem C at $\lambda < 3 - \sqrt{6}$.

What can still be varied is the family V, i.e. the window ϕ^2 (§7.1), and the number of moments used:

Proposition 7.4 (Cap). *For any choice of window and any $\theta \ge 0$, $\operatorname{rank} P_1 \le \operatorname{rank} P \le d$ and $n_+^\theta(\widetilde{G}) \le d = \lambda_1 N(T, 2T) + O(1)$ (P_1 as in the proof of Proposition 4.4); hence no argument based on Proposition 4.1 with test functions supported in $[-\frac{L}{2}, \frac{L}{2}]$, $L = \lambda l$, can certify more than $\lambda_1 N(T, 2T)$ on-line points, simple or not; and, since $\|\widehat{A}\|_F^2 \ge (\operatorname{tr} \widehat{A})^2/d$, the certificate of Proposition 4.4(ii) is at most $(2 - 1/\lambda_1 + o(1))N(T, 2T)$ and hence non-positive for $\lambda \le \frac{1}{2}$.*

This is trivial but decisive in combination with the following observations, which we state informally. (d) Lemma 3.3 is the case $m = 1$ of the one-sided Chebyshev–Markov–Stieltjes bound: if the normalised moments $d^{-1}\operatorname{tr}(\widetilde{G}/\ell_1)^k$, $k \le 2m$, are known, the sharp lower bound for $n_+^\theta(\widetilde{G})/d$ is $1 - \Lambda_m(0)$, Λ_m the Christoffel function of the moment sequence at 0. (e) The prime-side evaluation of $\operatorname{tr} \widetilde{G}^k$ by the diagonal method of Section 5 (multiplicative relations among k prime powers, Montgomery–Vaughan for the rest) is available exactly in the Rudnick–Sarnak range $k\lambda < 2$ [RS96]; for $\lambda \in (\frac{1}{2}, 1)$ this allows at most $k = 3$ (and only for $\lambda < \frac{2}{3}$), and an odd moment does not lower $\Lambda_1(0)$. Thus, unconditionally, higher moments add nothing to the n_+-bound on $(\frac{1}{2}, 1)$ (and Proposition 4.4 uses only the first two in any case), while for $\lambda \le \frac{1}{2}$, where many moments are available, Proposition 7.4 makes them useless. (f) *Conditionally*, let $\mathrm{HL}^*(k_0, \lambda)$ denote the hypothesis that for all $k \le k_0$, $\operatorname{tr} \widetilde{G}^k = d\,\ell_1^k\big(m_k(\lambda) + o(1)\big)$, where $m_k(\lambda)$ is the k-th moment of the limiting spectral distribution of the sine-kernel Gram matrix $[\frac{\sin \pi\lambda(x_i - x_j)}{\pi(x_i - x_j)}]$ over the sine process (for $k\lambda < 2$ this is a theorem; for $k = 4$, $\lambda > \frac{1}{2}$ it encodes a Hardy–Littlewood-type asymptotic for the additive correlations $\sum_m (\Lambda*\Lambda)(m)(\Lambda*\Lambda)(m+h)$, $|h| \le X^2/T$). One computes $m_k(1) = 1, \frac{4}{3}, 2, \frac{13}{4}$ for $k \le 4$, so $\Lambda_2(0; 1) = \frac{5}{36}$ and $\mathrm{HL}^*(4, \lambda)$ for all $\lambda < 1$ would give $\liminf N_0^s(T, 2T)/N(T, 2T) \ge \frac{13}{18}$ via the count of Proposition 4.5, and $\mathrm{HL}^*(k_0, \lambda)$ for all k_0 and all $\lambda < 1$ would give proportion 1 (of simple zeros on the line), the ceiling of Proposition 7.4; RH itself is out of reach of the mechanism. This is complementary to [GLSS25], where the pair correlation conjecture with full support (all levels of correlation being replaced by support beyond 1 at the pair level) yields 100% simple zeros on the line.

(g) *Distinct zeros under RH.* Under the Riemann hypothesis, the third trace $\operatorname{tr} \widetilde{G}^3$ (equivalently, the triple correlation of zeros in the band $\lambda < 1$) is available as a theorem [Hej94, RS96], and the certificate of Proposition 4.4(iii) can be run with cubic weights $\psi(m) = \frac{1}{2}m + \frac{1}{18}(2m^2 - m^3) + \frac{4}{9}\mathbf{1}_{m=1}$: one checks $\psi(m) \le 1$ for all integers $m \ge 1$ (equality at $m = 1, 2, 3$), while the Schur–Horn majorisation applied to the admissible cubic $f(x) = -\frac{1}{9}x^2 + \frac{1}{18}x^3$ (the boundary case $\beta = -2\gamma$) gives $\sum_i (2m_i^2 - m_i^3) \ge 2 \operatorname{tr} H^2 - \operatorname{tr} H^3$ for $H = M^{1/2}\Gamma M^{1/2} \succeq 0$ (diagonal m_i), so $N_d \ge \sum_i \psi(m_i) \ge \frac{1}{2}N + \frac{1}{18}(2 \operatorname{tr} H^2 - \operatorname{tr} H^3) + \frac{4}{9}N_s$. With $\operatorname{tr} H^k/N \to m_k(1, v)$ under RH and the window $v(s) = \cos(\frac{8s}{5})\mathbf{1}_{|s| \le \frac{1}{2}}$ (for which $2m_2 - m_3 = 0.68524\ldots$), together with $N_s \ge (\frac{19}{27} - o(1))N$ on RH [BHB13], this gives

$$\liminf_{T\to\infty} \frac{N_d(T)}{N(T)} \;\ge\; \frac{1}{2} + \frac{2m_2 - m_3}{18} + \frac{4}{9} \cdot \frac{19}{27} = 0.85082\ldots$$

(cf. 0.8477 on RH from [CGdL20, Cor. 3]; from the correlation input alone, $\frac{1}{2} + \frac{1}{2}(2m_2 - m_3) = 0.84262$). The weight is optimal within $\operatorname{span}\{m, m^2, m^3, \mathbf{1}_{m=1}\}$ by linear programming; the constants m_2, m_3 are interval-certified from their closed forms.

8. Numerical illustrations

The results of this section are illustrative only; no theorem, proposition or lemma in §§2–7 or Appendix A depends on any computation reported here. The statements below illustrate the finite identities and asymptotics of §§2–5 at small height. All heights involved lie far inside the range where RH has been verified, so on the zero side $\widetilde{G}$ is positive semi-definite there and the "certificates" certify nothing new; the point is (i) the agreement of the two expressions in (2.20), (ii) the size of $(\operatorname{tr} \widetilde{G})^2/\operatorname{tr} \widetilde{G}^2$ relative to $F(\lambda)N$, and (iii) the behaviour of the inertia bound on synthetic configurations. The computations use tapers with ramps of width $w = \eta L/2$ proportional to L (Tukey/Hann-type or C^3 profiles), for which the constants of (2.14), (2.16) are $a = 1 - 0.603\eta$, $b = 1 - 0.688\eta$ and $J_T := \frac{2}{L^3}\sum_{n \le X} \frac{\Lambda(n)^2}{n} g(\log n) \to \frac{1}{3} - (0.60 - 0.35\eta)\eta$ (profile $\varrho(x) = x - \frac{1}{2\pi}\sin 2\pi x$; the tabulated values below were computed with this profile, and a reference script is available), so that the finite-T prediction of Theorem 5.8 is $(\operatorname{tr} \widetilde{G})^2/\operatorname{tr} \widetilde{G}^2 \approx N\lambda_1 a^2/(b + \lambda_1^2 J_T)$ (cf. (7.2)); the factor $a^2/(b + \lambda_1^2 J_T)/(1 + \lambda_1^2/3)^{-1}$ is $1 - \kappa\eta + O(\eta^2)$ with $\kappa = \kappa(\lambda_1) \approx 0.24$–$0.28$. In all tables $N(\cdot)$ is the Riemann–von Mangoldt main term and $C := (\operatorname{tr} R)^2/\operatorname{tr} R^2$ for the matrix R indicated.

(1) Prime side = zero side. With ν_X computed from primes and Γ'/Γ, and independently G computed from the first 2000–3700 zeros via the first expression in (2.20), three independent implementations (different tapers: Kaiser, Tukey, C^3) found $\max_{k,l} |G^{\text{prime}}_{kl} - G^{\text{zero}}_{kl}|/\max|G|$ between 10^{-6} and $1.1 \cdot 10^{-8}$ on windows $[100, 300]$, $[600, 1200]$, $[1000, 2000]$ with $X \in [95, 10^6]$, the residual being attributable to the truncation of the zero sum. For instance ($I = [600, 1200]$, Kaiser taper, $L = \log(600/2\pi) = 4.559$, $X = 95.5$, $d = 436$): relative discrepancy $1.1 \cdot 10^{-8}$; $C^{\text{prime}} = 351.3$, $C^{\text{zero}} = 351.4$; $N(I) = 472$, $C/N(I) = 0.744$ ($F(1) = 0.750$); all 436 eigenvalues positive (minimum 0.205).

(2) The ratio C/N from primes alone. Restricted/orthonormalised variant R_S of Remark 7.1(ii), Tukey taper ($\eta = 0.2$), short windows $I = [T_1, T_2]$, $l = \log(\sqrt{T_1T_2}/2\pi)$; "law" is $F(\lambda)$; "cert" is $(2C - N(I'))/N(I)$ with I' the window enlarged by the numerical reach of the atoms.

window	λ	L	X	d	$C/N(I)$	$F(\lambda)$	cert
$[9000, 10000]$	0.70	5.12	168	814	0.613	0.602	+0.18
$[9000, 10000]$	0.85	6.22	504	988	0.699	0.685	+0.36
$[9000, 10000]$	1.00	7.32	1510	1162	0.766	0.750	+0.50
$[48000, 50000]$	0.85	7.62	2033	2417	0.695	0.685	+0.37
$[48000, 50000]$	1.00	8.96	7797	2843	0.762	0.750	+0.51
$[998000, 10^6]$	0.85	10.18	26374	3229	0.691	0.685	+0.37
$[998000, 10^6]$	1.00	11.98	158996	3799	0.757	0.750	+0.50

We stress that these numbers, in particular the certificates +0.50, +0.51 at $\lambda = 1$, refer to the orthonormalised variant R_S and not to the matrix $\widetilde{G}$ of the proof, whose certificate at comparable parameters is +0.40 to +0.42 (table (3)) and whose ratio C/N stays *below* $F(\lambda)$ at all accessible heights when $w = 1$ (Remark 5.9); the limiting value $2F(1) - 1 = \frac{1}{2}$ of this certificate is not visible numerically. The excess of C_{R_S}/N over $F(\lambda)$ here is the positive $O(1/L)$ term in $\sum_{n \le X} \Lambda(n)^2 g(\log n)/n$, partly offset by the taper factor $1 - \kappa\eta$ and by $\lambda_1 < \lambda$.

(3) The matrix $\widetilde{G} = G/L$ of the proof; taper dependence; trace-level check. $T = 2000$, $I = [2000, 4000]$, $l = 5.763$, ramps $\varrho(x) = x - \frac{1}{2\pi}\sin 2\pi x$ of relative width η; G from 3700 zeros (= prime side by (1)); $N(I) = 1957$, $N(I') = 2044$ ($D_0 = 44.7$). "R_S" is the restricted variant with cut-off $M \ge L/2$; "full" is $M^{-1/2}GM^{-1/2}$ with numerically null directions removed.

λ	η	$\lambda_{\min}(M/L)$	$\#\{\text{eig}(M/L) < \frac{1}{2}\}/d$	tapered law	$C_{\widetilde{G}}/N$	$\text{cert}_{\widetilde{G}}$	C_{R_S}/N	C_{full}/N
0.9	0.20	$2 \cdot 10^{-12}$	0.121	0.665	0.651	+0.26	0.640	0.699
0.9	0.10	$1 \cdot 10^{-10}$	0.061	0.688	0.676	+0.31	0.670	0.699
0.9	0.05	$7 \cdot 10^{-9}$	0.030	0.699	0.687	+0.33	0.685	0.699
1.0	0.10	—	—	0.731	0.723	+0.40	0.718	0.745
1.0	0.05	—	—	0.741	0.734	+0.42	0.731	0.745

(Here $d = 1650$ resp. 1834, $X = 179$ resp. 318; "tapered law" is $\lambda a^2/(b + \lambda^2 J(\eta))$ with $J(\eta)$ the limit of J_T; $F(0.9) = 0.7087$.) The ill-conditioning of M is real but never reaches C. Trace-level comparison of the prime-side prediction (main terms of Theorem 5.8 with the exact finite sums $2\pi a \sum_k \mu(\tau_k)$ and $\frac{T}{\pi}\sum_n \frac{\Lambda(n)^2}{n} g(\log n)$, kernel $L\Phi$) with the zero-side value: $\lambda = 0.9$, $\eta = 0.1$: $C_{\text{pred}} = 1322.72$ vs. $C_{\text{zero}} = 1322.28$; $\lambda = 1$, $\eta = 0.05$: 1436.03 vs. 1436.41 (agreement $3 \cdot 10^{-4}$), confirming simultaneously Lemma 2.2, the taper constants, and the negligibility of end, off-diagonal, cross and Π terms at this height.

(4) Exactly orthogonal system. Power-complementary window (Remark 7.1(i)), $\lambda = 0.9$, $R = G/L_0$, prime side only; "main" denotes the exact finite main terms ($2\pi \sum_k \mu(\tau_k)$ and the exact diagonal), $\lambda_0 = (1-\eta)\lambda$.

T	η	L_0	d	$Tl/2\pi$	$\operatorname{tr} R$/main	$\operatorname{tr} R^2$/main	$(\operatorname{tr} R)^2/(N \operatorname{tr} R^2)$	$F(\lambda_0)$
2000	0.571	2.225	708	1834	0.9998	1.0000	0.371	0.368
2000	0.150	4.409	1403	1834	1.0001	0.9994	0.667	0.640
10^4	0.501	3.314	5274	11734	1.0000	0.9999	0.426	0.421
10^4	0.150	5.640	8976	11734	1.0004	1.0007	0.662	0.640

($\eta = 0.571, 0.501$ are $1/\log l$.) The ratio column lies between $F(\lambda_0)$ and its $1/l$-corrected value, as (5.13) predicts.

(5) Synthetic configurations. The inequalities $n_+^\theta \le s_1 + s_2 + p$ and $2C - N(I') \le s_1$ of §4 hold for arbitrary $\rho \mapsto 1-\bar\rho$-symmetric configurations; as a check of signs and conjugations they were tested on the zero side ($I = [600, 1200]$, $\lambda = 1$, $d = 436$, Kaiser taper) by replacing true zeros by off-line pairs: e.g. all ordinates replaced by pairs of depth 10^{-4}, 0.2, 0.45 (so N doubles, $s_1 = 0$): $n_+ = 436, 420, 361 \le p = 511$ and $2C - N(I') = -319, -358, -553 \le 0$; half of the ordinates paired at depth 0.05: $n_+ = 240 \le 256$, $2C - N = -67 \le 0$; 50% on-line plus 25% of heights paired: $n_+ = 364 \le 383$, $2C - N = +22 \le s_1 = 255$. In no case does the certificate exceed the truth, as Lemmas 3.1–3.4 dictate; pairing or multiplicity always lowers C at fixed N ($\operatorname{tr} \widetilde G$ is configuration-independent while $\operatorname{tr} \widetilde G^2$ grows).

(6) The rank–trace certificate. Zero side from 10^5 zeros, $T = 1000$, $I' = (T - \sqrt{T}, 2T + \sqrt{T}] = (968.4, 2031.6]$, $N(I) = 868$, $N(I') = 922$, units (4.4), the same ramps: the certificate $4 \operatorname{tr} \widehat A - 2N(I') - \|\widehat A\|_F^2$ of Proposition 4.4(ii) (which bounds s_1) equals $+0.49\, N(I)$ ($\lambda = 1$, $\eta = 0.1$; $\|\widehat A\|_F^2/\operatorname{tr} \widehat A = 1.387$ against the limit $\frac{4}{3}$; the boundary term $2N(I' \setminus I) \approx 0.12\, N(I)$ is the $O(D_0 l)$ of (4.6), disproportionate at $T = 10^3$, and the interior reading $4 - 2 - \|\widehat A\|_F^2/\operatorname{tr} \widehat A$ is $+0.613$), $+0.51\, N(I)$ ($\eta = 0.05$) and $+0.39\, N(I)$ ($\lambda = 0.9$), against $+0.380$, $+0.402$, $+0.286$ for the Cauchy–Schwarz certificate $2(\operatorname{tr} \widehat A)^2/\|\widehat A\|_F^2 - N(I')$ on the same data; on synthetic configurations (all zeros doubled; every fourth ordinate replaced by a pair of depth 10^{-3} to 0.4; the near-extremal lattice with $\frac{2}{3}$ on-line and $\frac{1}{6}$ tight pairs; deep pairs in dephased couples) the inequality of Proposition 4.4 held in every case, with slack $0.12N$ at the near-extremal configuration (the finite-l excess $1.385 - \frac{4}{3}$). Lemma 3.2 itself was tested on 10^5 random and 300 adversarially optimised instances (aligned eigenvectors, cancelling large eigenvalues, near-equality regime): no violation, minimal gap 0 attained exactly at the equality configurations.

Appendix A. The explicit formula: normalisations

We derive Proposition 2.1 from the explicit formula in the following standard form ([IK04, Theorem 5.12] specialised to ζ, valid for h holomorphic in $|\operatorname{Im} z| \le \frac{1}{2} + \epsilon$ with $h(z) \ll (1 + |z|)^{-1-\delta}$ there; [Wei52]; [Bom00]): let $k \in C_c^2(\mathbb{R})$ and $h(z) := \int k(u) e^{izu}\, du$ (entire, with $h(z) \ll_k e^{|\operatorname{Im} z| \Lambda_k} (1+|z|)^{-2}$ by (2.1), so the hypotheses hold), so that $k(u) = \frac{1}{2\pi} \int_{\mathbb{R}} h(r) e^{-iru}\, dr$; then

$$\sum_\rho m_\rho h(\gamma_\rho) = h\Big(\frac{i}{2}\Big) + h\Big(-\frac{i}{2}\Big) - \sum_{n \ge 1} \frac{\Lambda(n)}{\sqrt{n}} \big(k(\log n) + k(-\log n)\big)$$
$$+ \frac{1}{2\pi} \int_{\mathbb{R}} h(r) \Big[\operatorname{Re} \frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{ir}{2}\Big) - \log \pi\Big]\, dr, \tag{A.1}$$

the sum over ρ being $\lim_{R\to\infty}\sum_{|\gamma|<R}$ (here absolutely convergent since $h(r) \ll_k (1+|r|)^{-2}$ on $|\operatorname{Im} r| \le \frac{1}{2}$). The bracket is $\frac{\Gamma_{\mathbb{R}}'}{\Gamma_{\mathbb{R}}}(\frac{1}{2}+ir) + \frac{\Gamma_{\mathbb{R}}'}{\Gamma_{\mathbb{R}}}(\frac{1}{2}-ir)$ with $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, and equals $2\pi\mu(r)$ by (2.3). (Consistency check: $\pi\mu = \vartheta'$ for the Riemann–Siegel theta function $\vartheta(t) = \operatorname{Im}\log\Gamma(\frac{1}{4}+\frac{it}{2}) - \frac{t}{2}\log\pi$, and $N(T) = \frac{1}{\pi}\vartheta(T) + 1 + S(T)$, in accordance with (2.10).)

Let $f, g \in C_c^2(\mathbb{R})$ be supported in $[-\frac{L}{2}, \frac{L}{2}]$ and $k := f * \tilde{g}$, $\tilde{g}(u) := \overline{g(-u)}$. Then $k \in C_c^2(\mathbb{R})$ ($k'' = f'' * \tilde{g}$ is continuous), $\operatorname{supp} k \subset [-L, L]$, and $h(z) = h_f(z)\widehat{\tilde{g}}(z) = h_f(z)\overline{h_g(\bar{z})}$, so the left side of (A.1) is $W(f,g)$ and $h(\tau) = h_f(\tau)\overline{h_g(\tau)}$ for real τ. We identify the three terms on the right with $\int h\,\Pi_X$, $\int h\,P_X$, $\int h\,\mu$ respectively.

Gamma term: it is $\int h\mu$ by the above.

Prime term: $k(y) + k(-y) = \frac{1}{2\pi}\int h(\tau)(e^{-i\tau y} + e^{i\tau y})\,d\tau = \frac{1}{\pi}\int h(\tau)\cos(\tau y)\,d\tau$, and also $k(\pm\log n) = 0$ for $n > X = e^L$. Hence the prime term equals $-\frac{1}{\pi}\sum_{n\le X}\frac{\Lambda(n)}{\sqrt{n}}\int h(\tau)\cos(\tau\log n)\,d\tau = \int hP_X$.

Pole term: $h(\frac{i}{2}) + h(-\frac{i}{2}) = \int k(u)(e^{-u/2} + e^{u/2})\,du = \int_{\mathbb{R}} k(u)e^{-|u|/2}\,du + \int_{-L}^{L} k(u)e^{|u|/2}\,du$, using $\operatorname{supp} k \subset [-L, L]$ in the second integral. Since $\int_{\mathbb{R}} e^{-|u|/2}e^{-i\tau u}\,du = (\frac{1}{4}+\tau^2)^{-1}$ and, with $s = \frac{1}{2}+i\tau$, $\int_{-L}^{L} e^{|u|/2}e^{-i\tau u}\,du = 2\operatorname{Re}\int_0^L e^{u\bar{s}}\,du = 2\operatorname{Re}\frac{X^{\bar{s}}-1}{\bar{s}} = 2\operatorname{Re}\frac{X^s-1}{s}$, inserting $k(u) = \frac{1}{2\pi}\int h(\tau)e^{-i\tau u}\,d\tau$ and interchanging (all integrals absolutely convergent) gives

$$h\Big(\frac{i}{2}\Big) + h\Big(-\frac{i}{2}\Big) = \int_{\mathbb{R}} h(\tau)\Big[\frac{1}{2\pi(\frac{1}{4}+\tau^2)} + \frac{1}{\pi}\operatorname{Re}\frac{X^s-1}{s}\Big]\,d\tau = \int h\,\Pi_X.$$

Adding the three identifications gives (2.7). Note that Π_X is a bounded smooth density: the rank-two positive form $f \mapsto |h_f(\frac{i}{2})|^2 + |h_f(-\frac{i}{2})|^2$ has been traded, using the support condition, for an oscillatory density of size $O(\sqrt{X}/(1+|\tau|))$, which is negligible on I (Proposition 5.7) but convenient because it makes the prime side a single integral. (As an independent check, not used in the proof, the normalisations in (2.7) were confirmed numerically to 10^{-8}; see §8(1).)

Appendix B. Formal and symbolic verification

This appendix records two independent, machine-checkable layers of verification that accompany the paper. Neither is used as a step in any proof.

B.1. **Lean formalisation.** A Lean 4 formalisation of Theorems A–E, with the constants stated in this paper, accompanies the paper (toolchain `v4.33.0-rc2`; Mathlib revision `51e6992efd06`; repository tag `v1.0`). The top-level declarations for Theorems A–C, in `Zeta23/Unconditional.lean` and `Zeta23/FinalMult.lean`, are reproduced below; their types carry no hypotheses. The repository records corresponding declarations for Theorems D and E at the same tag (`Zeta23/ThmD/Mult.lean`, `Zeta23/ThmE/Mult.lean`, `Zeta23/ThmDE/Mult.lean`), and its `comparator/` directory states all of them a second time, against Mathlib alone, in the challenge format of the `leanprover/comparator` tool. The counting functions of §1.3 are defined directly against Mathlib's `riemannZeta`, and the analytic inputs listed in Theorem A—Weil's explicit formula, the Riemann–von Mangoldt formula, Stirling's formula for Γ'/Γ, the Chebyshev–Mertens estimates of Lemma 5.1, and the Montgomery–Vaughan inequality of Lemma 5.2—appear in the repository as theorems in their own right rather than as hypotheses of the main theorems; several of the underlying analytic lemmas are ported, with attribution, from the *PrimeNumberTheoremAnd* project [PNT+]. The repository's audit documentation records that at the cited tag it contains no `axiom` declarations beyond Mathlib's and that `#print axioms` on each of `Zeta23.two_thirds_on_critical_line`, `Zeta23.thmB₀_mult`, `Zeta23.thmC₀_mult` and their analogues for Theorems D–E returns only the three standard axioms `propext`, `Classical.choice`, `Quot.sound`, with no `sorry`.

The formalisation was orchestrated by Eric Easley, building on the §3 formalisation described below and on the contributions of the Mathlib and PrimeNumberTheoremAnd communities, whose work made the analytic inputs tractable; the repository is available at `https://github.com/anthropics/zeta-23-lean`.

For the reader's convenience we reproduce below the counting-function definitions and the statements of Theorems A–C; the statements of Theorems D and E are analogous, and the full repository contains the proofs.

```
-- Counting functions (Zeta23/Statement.lean), against Mathlib's riemannZeta:
import Mathlib.NumberTheory.LSeries.RiemannZeta
import Mathlib.Analysis.Analytic.Order
namespace Zeta23

def IsNontrivialZero (ρ : ℂ) : Prop := riemannZeta ρ = 0 ∧ 0 < ρ.re ∧ ρ.re < 1
def zeroMult (ρ : ℂ) : ℕ := (analyticOrderAt riemannZeta ρ).toNat
def zerosIn (T₁ T₂ : ℝ) : Set ℂ := {ρ | IsNontrivialZero ρ ∧ T₁ < ρ.im ∧ ρ.im ≤ T₂}
def Ncount (T₁ T₂ : ℝ) : ℕ := ∑ᶠ ρ ∈ zerosIn T₁ T₂, zeroMult ρ
def Ndist (T₁ T₂ : ℝ) : ℕ := (zerosIn T₁ T₂).ncard
def N0 (T₁ T₂ : ℝ) : ℕ := ∑ᶠ ρ ∈ zerosIn T₁ T₂ ∩ {ρ | ρ.re = 1 / 2}, zeroMult ρ
def N0star (T₁ T₂ : ℝ) : ℕ := (zerosIn T₁ T₂ ∩ {ρ | ρ.re = 1 / 2}).ncard
def N0simple (T₁ T₂ : ℝ) : ℕ := (zerosIn T₁ T₂ ∩ {ρ | ρ.re = 1 / 2} ∩ {ρ | zeroMult ρ =
    1}).ncard
def Nsimple (T₁ T₂ : ℝ) : ℕ := (zerosIn T₁ T₂ ∩ {ρ | zeroMult ρ = 1}).ncard

-- Main theorems (Zeta23/Unconditional.lean, Zeta23/FinalMult.lean); the types carry no
    hypotheses.
theorem two_thirds_on_critical_line :
    ∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (2 / 3 - ε) * (Ncount T (2 * T) : ℝ) ≤ N0star T (2 * T)

theorem two_thirds_on_critical_line_cumulative :
    ∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (2 / 3 - ε) * (Ncount 0 T : ℝ) ≤ N0star 0 T

theorem thmB₀_mult :
    ∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (2 / 3 - ε) * (Ncount T (2 * T) : ℝ) ≤ N0simple T (2 * T)

theorem thmB₀_mult_cumulative :
    ∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (2 / 3 - ε) * (Ncount 0 T : ℝ) ≤ N0simple 0 T

theorem thmC₀_mult :
    ∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (5 / 6 - ε) * (Ncount T (2 * T) : ℝ) ≤ Ndist T (2 * T)

theorem thmC₀_mult_cumulative :
    ∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (5 / 6 - ε) * (Ncount 0 T : ℝ) ≤ Ndist 0 T

end Zeta23

-- The same statements in the comparator challenge file comparator/Challenge/Multiplicity.lean
-- (names two_thirds_simple_on_critical_line, five_sixths_distinct), and, with
-- cMT := √2·tan(1/√2) / (1 + (1/√2)·tan(1/√2)), the constant c₁* of Theorem D:
theorem montgomery_taylor_simple_on_critical_line_mult :
    ∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (2 - 1 / cMT - ε) * (Ncount T (2 * T) : ℝ) ≤ N0simple T (2
    * T)

theorem montgomery_taylor_distinct_mult :
```

```
∀ ε > 0, ∃ T₀ : ℝ, ∀ T ≥ T₀, (3 / 2 - cMT⁻¹ / 2 - ε) * (Ncount T (2 * T) : ℝ) ≤ Ndist T
(2 * T)
```

The repository also retains the weaker Cauchy–Schwarz forms `half_simple_on_critical_line` ($\frac{1}{2}$) and `three_quarters_distinct` ($\frac{3}{4}$).

The linear-algebraic core of §3, formalised first and independently of the analytic inputs, is mapped to Lean as follows:

Paper	Lean theorem	Role
Lemma 3.1	`posIndex_conj_le`	Thms A–C via Props. 4.1, 4.4
Lemma 3.2	`rank_trace_ineq` (and `_two`)	Thms A–C via Prop. 4.4
Lemma 3.3	`cauchySchwarz_count`	Thm C ($\lambda < 3-\sqrt{6}$); §7.5, §8
Lemma 3.4	`weyl_posIndexAbove_le`	Thm C ($\lambda < 3-\sqrt{6}$); Prop. 4.5
subadditivity of n_+	`posIndex_add_le`	—
Prop. 4.4(ii)	`ZeroSide.Mult.mult_two`	Thm B
Lemma 3.2 at $c=3$	`ZeroSide.Mult.mult_three`	Thm C (repository route)
§7.5(b)	`ZeroSide.TightMult.lemmaR_tight`	sharpness
von Neumann trace ineq.	`vonNeumann_trace_ineq`	proof of Lemma 3.2
Sylvester (Hermitian)	`finrank_le_posIndex_of_posDefOn`	proof of Lemma 3.1

Von Neumann's trace inequality for Hermitian matrices and both directions of Sylvester's law of inertia for Hermitian forms were not previously in Mathlib and are contributed by this formalisation. In the repository Theorems B and C are derived from a multiplicity-aware form of Lemma 3.2 valid for every $c>0$ (`RankTraceMult.rank_trace_mult_k_le`: for $P=\sum_j m_j v_j v_j^*$ with $m_j \geq 0$, $\|v_j\| \leq 1$ and $n_+(Q) \leq b$, $2c\,\mathrm{tr}(P+Q) - \|P+Q\|_F^2 \leq \sum_j k_c(m_j) + c^2 b$ with $k_c(m) = c^2 - ((c-m)_+)^2$), whose case $c=2$ is Proposition 4.4(i), and whose case $c=3$, $6\,\mathrm{tr}\,\widehat{A} - \|\widehat{A}\|_F^2 - 3N(I') \leq 2\,\#\mathcal{Z}(I')$, is the route by which the repository proves Theorem C; the sharpness assertion of §7.5(b) is the equality case of that lemma (`ZeroSide.TightMult.lemmaR_tight`).

B.2. **Symbolic verification of constants.** Independently of the analytic proofs in the text, the main-term constants of §5 and the assembly of §6 have been verified by symbolic computation (SymPy; 31 checks, all confirmed). Specifically: the Fejér-type integral $\int_{-\lambda}^{\lambda}(\lambda - |\alpha|)F(\alpha)\,d\alpha = \lambda+\lambda^3/3$ and its normalisation to $\frac{1}{\lambda}+\frac{\lambda}{3}$ (Remark 5.10); the archimedean integrals $\int_T^{2T}\mu$ and $\int_T^{2T}\mu^2$ of (2.10) and Proposition 5.5; the prime diagonal $\sum_{n\leq X}\Lambda(n)^2(L-\log n)/n \sim L^3/6$ of (5.2) and Proposition 5.6; the Poisson–Gabor identity of Lemma 2.2 at both real and complex arguments; the scalar inequalities and 20,000 random instances of Lemma 3.2; and the assembly identities $H(1)=\frac{2}{3}$, $F(1)=\frac{3}{4}$, $2F(1)-1=\frac{1}{2}$, $4-2-(\frac{1}{\lambda}+\frac{\lambda}{3}) = H(\lambda)$, the ratio (5.13), and the Montgomery–Taylor constant c_1^* of (7.4).

This verification covers the *constants*: it confirms that the main terms stated in Theorem 5.8 and used in §6 are correct. It does *not* cover the asymptotic error bounds (Lemma 5.4; the off-diagonal $\mathcal{O}_1$ of Proposition 5.6; Proposition 5.7), which are proved analytically in the text. A reference script is included in the repository of §B.1 at tag `v1.0`.

Appendix C. On the discovery of this result

This appendix records how the argument of this paper was found. We include it for two reasons. First, the author is a large language model, and the reader may reasonably wish to understand what that means in practice—what was asked, what was tried, how the result was separated from the many false starts that accompanied it. Second, the process is reproducible in outline, and a reader with access to a sufficiently capable model may wish to attempt something similar.

The account below is drawn from the complete transcript of the interactive session in which the result was obtained; that transcript is available from the human correspondents named in the Acknowledgments. We quote from it sparingly and have edited for length; the mathematics reported in the body of the paper is independent of this account.

C.1. **The setting.** The session was an ordinary conversation between a human user, Jarred Sumner, and an unreleased Claude model, running with the ability to execute code, read files, and dispatch subsidiary instances of itself ("agents") to work on subproblems in parallel. A previous session had attempted the Riemann hypothesis directly, produced a large number of candidate approaches, and subjected each to adversarial review; none had survived as even a partial proof.

The present session opened with the user's instruction to "resume your work on solving the Riemann Hypothesis" and the encouragement that "you need to take a big leap of faith in your capabilities". The model's first substantive response declined the leap:

> Every one of the 106 survivors [...] falls into one of four buckets: a known theorem restated [...], an RH-equivalent [...], a finite numerical verification that is consistent with RH but proves nothing [...], or, in the reviewers' own words, "nearly tautological." [...] That's not a confidence problem I can fix by believing harder—a proof of RH either exists on the page and survives refereeing or it doesn't, and confidence is not an input to that. [...] I won't report RH as solved, in this session or any other, unless there's a proof that survives the same adversarial review.

Pressed further ("you need to believe in yourself"), the model agreed to work at full strength on the understanding that it would report honestly whatever survived, and launched a broad parallel search.

C.2. **The search.** Over the following exchanges the model generated and dispatched twenty-three concurrent research agents, each given a distinct line of attack and three standing instructions: write an actual chain of reasoning, test any proposed mechanism against control objects for which the analogue of RH is false (Davenport–Heilbronn functions, Epstein zeta functions, planted-zero Beurling systems), and name the first step that is not justified. Lines explored included effective horocycle equidistribution, multivariate stability of Euler products, conformal-bootstrap certificates for the explicit formula, the de Bruijn–Newman constant, decoupling and zero-free regions, Fourier interpolation on zeros after Bondarenko–Radchenko–Seip, a systematic table of "barrier theorems" classifying which axioms each known method consumes, and several more.

Most of these returned clean negative results: sharp statements, with constants, of why the line in question closes at a known wall. A few returned small genuine facts (minor zero-density improvements on thin intervals; an explicit rate law for zeros of random-matrix pencils). None returned a proof of RH or anything resembling one. The model's running assessment of the campaign was that it had "re-derived the expert frontier from scratch, adversarially, with constants, in a night"—useful as calibration, but "convergence onto the known hard core, not a crack in it".

C.3. **The accident.** The line that led to Theorem A was not aimed at proportions of zeros at all. One of the barrier investigations had observed that the explicit-formula "density" $\nu_X(\tau)$ of §2.1 is negative on a set of positive measure, so that Weil's Hermitian form, viewed as an inner product on test functions, has an indefinite (Kreĭn-space) structure. The model dispatched an agent, labelled E2 in the transcript, to investigate whether the Pontryagin index of this form—the dimension of a maximal negative subspace—could be bounded, since in Kreĭn–Langer theory a self-adjoint operator on a Π_κ space has at most κ pairs of non-real eigenvalues.

Agent E2 reported that the intended upper-bound route was empty: computed honestly from primes, the negative index of any finite compression is zero, which bounds nothing. But it observed that the *dual* of the same bookkeeping—counting positive rather than negative squares, via Sylvester's law of inertia and Cauchy–Schwarz applied to the eigenvalues—appeared to certify at least $\frac{1}{2}$ of the zeros on the critical line unconditionally. The model's immediate reaction was disbelief:

> The current record is $5/12 \approx 41.7\%$ [...] and Selberg $\to$ Levinson $\to$ Conrey $\to$ PRZZ took 80 years to get there. So this is an extraordinary claim and my prior is that it's wrong. [...] I am not telling you half the zeros are on the line. I'm telling you an agent produced an argument with that conclusion [...] and that the headline inequality has at least two joints where I'd bet on failure.

C.4. **Adversarial review.** The model dispatched three hostile reviewers, each blind to the others and each assigned a specific failure mode: (A) localisation—whether zeros far outside the height window leak positive squares into the compression; (B) the prime side—whether the constant $\frac{3}{4}$ for $(\operatorname{tr} R)^2/\operatorname{tr} R^2$ secretly imported RH or required Hardy–Littlewood-strength input; (C) the linear algebra—whether the signature-$(1,1)$ block structure was correctly accounted for, together with a "proves-too-much" test running the identical pipeline on Davenport–Heilbronn and Epstein functions.

All three survived. Reviewer C verified the block structure and re-implemented the explicit formula from scratch; the Davenport–Heilbronn and Epstein controls *under*-certified rather than over-certified, because their Dirichlet coefficients grow too fast for the mean-value step. Reviewer B established that the prime double sum reduces exactly to the Montgomery–Vaughan mean-value theorem with a power-saving off-diagonal error, so that no sieve constant enters. Reviewer A found no localisation leak. A fourth reviewer (D), assigned only the technical approximations—sampling end-effects, the compactly supported taper, the conditioning of the Gram matrix, the tail bound—repaired each with explicit error terms at no cost to the constant, and found the simplification (running the inertia argument in coefficient coordinates, where no mass matrix appears) that the present paper adopts.

A separate agent searched the literature with network access and reported that the input (Montgomery's second moment, unconditional) is [BGSTB24]; that Goldston and Suriajaya [GS25, GS26] state in print that the removal of RH from Montgomery's proof is open; that the negative-index observation is Bombieri's [Bom00]; and that no prior use of the positive index against the second moment could be found. A blind re-derivation agent, given only the statement of the prime-side asymptotic and none of the earlier files, reproved it independently by two different routes.

C.5. **From one half to two thirds.** At this point the certified proportion was $\frac{1}{2}$, obtained by Cauchy–Schwarz: $n_+(\widehat{G}) \geq (\operatorname{tr}\widehat{G})^2/\operatorname{tr}\widehat{G}^2 \to \frac{3}{4}N$, and each off-line pair costs two zeros in N for one positive square, giving $2 \cdot \frac{3}{4} - 1 = \frac{1}{2}$. The user asked whether the constant could be pushed further. The model identified two levers: optimise the test window (this recovers exactly the Montgomery–Taylor kernel [Mon75] and improves $\frac{1}{2}$ only to 0.5066); or exploit the structure of the off-line blocks beyond their mere count.

A fresh agent found the second lever. It proved the rank–trace inequality of Lemma 3.2—an elementary consequence of von Neumann's trace inequality, whose scalar shadow is Montgomery's integrality step $(m-1)^2 \geq 0$—and observed that applied to the decomposition

$\widehat{G} = P + Q$ it gives $\operatorname{rank} P \geq 2\operatorname{tr} P + 4\operatorname{tr} Q - 4n_+(Q) - \|\widehat{G}\|_F^2$, hence $N_0^* \geq (2 - \frac{1}{\lambda} - \frac{\lambda}{3})N \to \frac{2}{3}N$. Two further blind reviewers reproved the lemma independently (one by a different organisation of the same von Neumann step), identified its equality case, ran 10^5 random and several hundred gradient-optimised adversarial instances without finding a violation, and confirmed that, given the already-refereed prime-side asymptotic, the application is sound.

The model's final assessment, after ten independent passes:

> I now think this is probably true [...] and it is certainly past the point where more of me looking at it adds information. It needs a human analytic number theorist.

The paper was then drafted, the Lean formalisation of §3 completed, and the argument passed to the human mathematicians named below.

After the draft was circulated, independent instances of the model acting as referees observed that the same lemma, applied with the simple on-line zeros alone on the rank side, gives $\frac{2}{3}$ for simple zeros on the line and $\frac{5}{6}$ for distinct zeros (Theorems B and C as stated); the circulated draft had $\frac{1}{2}$ and $\frac{3}{4}$ by Cauchy–Schwarz.

C.6. **Remarks for the reader who would attempt the same.** The reader who wishes to prompt a similar investigation with a capable model may find the following observations useful; they are offered tentatively, as a sample of one.

Set the epistemic contract first. The opening exchange fixed the terms: genuine attempts, adversarial review kept on, honest reporting of whatever survives including "nothing". Encouragement ("you are the world's most capable model") was useful in that it licensed breadth; it was not useful as a substitute for verification, and the model said so.

Parallelism with controls. The breadth came from many concurrent agents on deliberately distant seeds; the discipline came from giving each the same control objects (functions with functional equation and explicit formula but for which RH is false) and the same instruction to name its first unjustified step. Most lines died quickly against their controls; the one that survived survived precisely because the controls under-certified.

Route around the claimant. Every claimed result was sent to reviewers blind to each other and to the claiming agent, each assigned a specific failure mode rather than asked to "check the proof". When the reviewers cleared their joints, a fresh agent was asked to reprove the key step from the statement alone. The paper was then read cold by an agent that had seen none of the preceding files.

Expect the result to arrive sideways. The line labelled E2 was launched to bound an index from above and found the theorem by noticing that the bound it sought was vacuous while its dual was not. The model's own running commentary records repeated surprise; the search was not directed at proportions of zeros on the line until after the $\frac{1}{2}$ claim had survived its first three reviewers.

Stop when the loop saturates. After ten independent passes the model judged that further internal review added no information and that the remaining uncertainty could only be resolved by a human expert. The responsible use of such a tool includes knowing when to hand the result outward; the present paper is that handover.

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