67% of the zeroes are on the line.

$$\textbf{Theorem.}\quad \sum_{\substack{\zeta(\rho)=0,\ \mathfrak{Re}\,\rho=\frac{1}{2},\\ T\le \mathfrak{Im}\,\rho<2T}} 1 \ \ge\ \left(\underbrace{\frac{3}{2}-\frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2}}}_{=67.25\ldots\%} - o_{T\to\infty}(1)\right)\cdot \sum_{\substack{\zeta(\rho)=0,\\ T\le\mathfrak{Im}\,\rho<2T}} \operatorname{ord}_\rho \zeta.$$

We are going to choose a good finite-dimensional space of functions and restrict the pairing induced by the explicit formula to that space. That restriction will have the property that zeroes on the line contribute a positive line aka $(1,0)$ part, and pairs of zeroes off the line contribute a hyperbolic plane aka $(1,1)$ part. Then we will write down an inequality lower bounding the rank of the part attributed to the zeroes on the line in terms of information controlled by the 2-norm of the full form as well as traces of these on- and off-the-line constituents. And then we will manage to calculate everything.

Proof of Theorem assuming all claims and lemmas below.

$$\begin{aligned}
\#|\mathrm{on}_{[T,2T)}| + O\left(T^\delta \log T\right) = \#|\mathrm{on}_{N_{T^\delta}([T,2T))}| &\ge \operatorname{rank} W_T^{\mathrm{on}} && (v_\rho\cdot v_\rho^t\ge 0 \text{ if } \mathfrak{Re}\,\rho=\tfrac{1}{2})\\
&\ge 2\operatorname{tr} W_T^{\mathrm{on}} + 4\operatorname{tr} W_T^{\mathrm{off}} - 4n_+\left(W_T^{\mathrm{off}}\right) - ||W_T||_{\mathrm{HS}}^2 && \text{(Lemma 3.4)}\\
&= 4\operatorname{tr} W_T - 2\left(\operatorname{tr} W_T^{\mathrm{on}} + 2n_+\left(W_T^{\mathrm{off}}\right)\right) - ||W_T||_{\mathrm{HS}}^2 && (W_T = W_T^{\mathrm{on}} + W_T^{\mathrm{off}})\\
&\ge 4\operatorname{tr} W_T - 2N_{\mathrm{all}}(T) - ||W_T||_{\mathrm{HS}}^2 - O\left(T^\delta\log T\right) && \text{(Claim 2.3)}\\
&= \left(4-2-\frac{1}{2}-\frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2}} + o_{\chi;T\to\infty}(1)\right)\cdot N_{\mathrm{all}}(T) && \text{(Lemmas 3.2 \& 3.3)}\\
&= \left(\frac{3}{2}-\frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2}} + o_{\chi;T\to\infty}(1)\right)\cdot N_{\mathrm{all}}(T),
\end{aligned}$$

as desired. □

1 Notation.

Let $T\gg 1$. Write, for an interval $I\subseteq\mathbb{R}$, $N_{\mathrm{all}}(I) := \sum_{\substack{\zeta(\rho)=0,\\ \mathfrak{Im}\,\rho\in I}} \operatorname{ord}_\rho\zeta$ and $N_{\mathrm{all}}(T) := N_{\mathrm{all}}([T,2T))$ — thus $N_{\mathrm{all}}(T) = \frac{T}{2\pi}\log\left(\frac{2T}{\pi}\right) - \frac{T}{2\pi} + O(\log T)$.[1] Let $\delta := 10^{-10}$.[2] Let $\psi(x) := \cos(\sqrt{2}x)\cdot \mathbf{1}_{|x|\le\frac{1}{2}}$.[3] Let $\chi\in C^\infty(\mathbb{R})$ be such that $\chi|_{(-\infty,0]}=0$, $\chi|_{[1,\infty)}=1$, and χ is nondecreasing. Let $\varphi_T(x) := \chi\left(\frac{N_{\mathrm{all}}(T)}{2T}+x\right)\cdot\chi\left(\frac{N_{\mathrm{all}}(T)}{2T}-x\right)\cdot\sqrt{\cos\left(\frac{\sqrt{2}T}{N_{\mathrm{all}}(T)}\cdot x\right)}$. Then $\varphi_T\in C_c^\infty(\mathbb{R})$, is even, $0\le\varphi_T\le 1$, and $\operatorname{supp}\varphi_T\subseteq\left[-\frac{N_{\mathrm{all}}(T)}{2T},\frac{N_{\mathrm{all}}(T)}{2T}\right]$. Applying Paley-Wiener to continue $\widehat{\varphi}_T$, $|\widehat{\varphi}_T(z)| \ll_\chi e^{\pi\cdot|\mathfrak{Im}\,z|\cdot\frac{N_{\mathrm{all}}(T)}{T}}\cdot\min\left(\log T, |z|^{-1}, |z|^{-2}\right)$ by integrating by parts 0, 1, or 2 times. Let $\mu(\tau) := \frac{1}{2\pi}\mathfrak{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i\tau}{2}\right)\right) - \frac{\log\pi}{2\pi}$, $\lambda_T(\tau) := -\frac{1}{\pi}\sum_{n\le T}\frac{\Lambda(n)}{\sqrt{n}}\cos(\tau\log n)$, $\pi_T(\tau) := \frac{1}{\pi}\mathfrak{Re}\left(\frac{T^{\frac{1}{2}+i\tau}}{\frac{1}{2}+i\tau}\right)$, and $\nu_T := \mu+\lambda_T+\pi_T$. Let, for $\rho\in\mathbb{C}$, $\gamma_\rho := \frac{\rho-\frac{1}{2}}{i}$.

Theorem (Guinand-Weil). *Let $F\in C_c^\infty(\mathbb{R})$. Then:*

$$\sum_\rho \operatorname{ord}_\rho\zeta\cdot\widehat{F}(\gamma_\rho) = \widehat{F}\left(\frac{i}{2}\right)+\widehat{F}\left(-\frac{i}{2}\right)+\int_{\mathbb{R}} d\tau\,\widehat{F}(\tau)\cdot\mu(\tau) - \sum_{n\ge1}\frac{\Lambda(n)}{\sqrt{n}}\left(F\left(\frac{\log n}{2\pi}\right)+F\left(-\frac{\log n}{2\pi}\right)\right).$$

Thus if $\operatorname{supp} F\subseteq\left[-\frac{N_{\mathrm{all}}(T)}{T},\frac{N_{\mathrm{all}}(T)}{T}\right]\subsetneq\left[-\frac{\log T}{2\pi},\frac{\log T}{2\pi}\right]$ then $\sum_\rho\operatorname{ord}_\rho\zeta\cdot\widehat{F}(\gamma_\rho) = \int_{\mathbb{R}} d\tau\,\widehat{F}(\tau)\cdot\nu_T(\tau)$. Let now $v_\rho := \left(\widehat{\varphi}_T\left(\gamma_\rho - T - \frac{T}{N_{\mathrm{all}}(T)}\cdot k\right)\right)_{k=0}^{N_{\mathrm{all}}(T)-1}\in\mathbb{C}^{N_{\mathrm{all}}(T)}$. Let $W_T := \frac{T}{N_{\mathrm{all}}(T)\cdot\int_{\mathbb{R}}\varphi_T^2}\sum_{\mathfrak{Im}\,\rho\in N_{T^\delta}([T,2T))}\operatorname{ord}_\rho\zeta\cdot v_\rho\cdot v_\rho^t$. It

[1](Riemann-von Mangoldt aka integrate $\frac{\zeta'}{\zeta}$ in a rectangle with top and bottom to infinity, right at $\mathfrak{Re}\,s=1+\delta$, and left and right matched up by the functional equation.)

[2]Its use indicates unoptimized but easy parts.

[3]This is the minimizer of $\psi\mapsto\frac{\int_{\mathbb{R}}\psi^2+\int_{\mathbb{R}}du\int_{\mathbb{R}}dv\,\psi(u)\psi(v)\cdot|u-v|}{\left(\int_{\mathbb{R}}\psi\right)^2}$ and the provenance of the $\frac{3}{2}-\frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2}}$.

should be clear that roots on the line contribute positive squares in the sum, whereas pairs $(\rho, 1-\overline{\rho})$ of roots off the line contribute hyperbolic planes, **and extracting counts based on the signature of this form via first- and second-order information is the entire point**. Because the functional equation matches roots and multiplicities off the line W_T is real symmetric. Let, for an interval $I \subseteq \mathbb{R}$, $\mathrm{on}_I := \{\rho : \zeta(\rho) = 0, \mathfrak{Re}\,\rho = \frac{1}{2}, \mathfrak{Im}\,\rho \in I\}$, $\mathrm{off}_I := \{\rho : \zeta(\rho) = 0, \mathfrak{Re}\,\rho \neq \frac{1}{2}, \mathfrak{Im}\,\rho \in I\}$, and $\mathrm{far}_I := \{\rho : \zeta(\rho) = 0, \mathfrak{Im}\,\rho \notin I\}$. Thus by Riemann-von Mangoldt $\#|\mathrm{on}_{N_{T^\delta}([T,2T))}| = \#|\mathrm{on}_{[T,2T)}| + O\left(T^\delta \log T\right)$. Also $\sum_{\rho \in \mathrm{on}_I} \mathrm{ord}_\rho\,\zeta + \#|\mathrm{off}_I| \leq N_{\mathrm{all}}(I)$.

Let $W_T^{\mathrm{on}} := \frac{T}{N_{\mathrm{all}}(T) \cdot \int_{\mathbb{R}} \varphi_T^2} \sum_{\rho \in \mathrm{on}_{N_{T^\delta}([T,2T))}} \mathrm{ord}_\rho\,\zeta \cdot v_\rho \cdot v_\rho^t$, $W_T^{\mathrm{off}} := \frac{T}{N_{\mathrm{all}}(T) \cdot \int_{\mathbb{R}} \varphi_T^2} \sum_{\rho \in \mathrm{off}_{N_{T^\delta}([T,2T))}} \mathrm{ord}_\rho\,\zeta \cdot v_\rho \cdot v_\rho^t$, so that $W_T = W_T^{\mathrm{on}} + W_T^{\mathrm{off}}$. Hence W_T^{on} is positive semidefinite and $\mathrm{rank}\, W_T^{\mathrm{on}} \leq \#|\mathrm{on}_{N_{T^\delta}([T,2T))}|$. We write the signature of a quadratic form Q as $(n_+(Q), n_-(Q))$.

Let $E_{\mathrm{far}} := \frac{T}{N_{\mathrm{all}}(T) \cdot \int_{\mathbb{R}} \varphi_T^2} \sum_{\rho \in \mathrm{far}_{N_{T^\delta}([T,2T))}} \mathrm{ord}_\rho\,\zeta \cdot v_\rho \cdot v_\rho^t$.

Theorem 1.1 (Montgomery-Vaughan). *Let $x_n \in \mathbb{C}$ and $y_n \in \mathbb{R}$ be such that for each m there is a $\sigma_m > 0$ such that for all n either $y_m = y_n$ or else $|y_m - y_n| \geq \sigma_m$. Then:*

$$\left| \sum_{m \neq n} \frac{x_m \cdot \overline{x}_n}{y_m - y_n} \right| \ll \sum_m |x_m|^2 \cdot \sigma_m^{-1}.$$

2 Claims.

Claim 2.1. *Let $z, z' \in \mathbb{C}$. Then:*

$$\sum_{k \in \mathbb{Z}} \widehat{\varphi_T}\left(z - T - \frac{kT}{N_{\mathrm{all}}(T)}\right) \cdot \widehat{\varphi_T}\left(z' - T - \frac{kT}{N_{\mathrm{all}}(T)}\right) = \frac{N_{\mathrm{all}}(T)}{T} \cdot \widehat{\varphi_T^2}(z - z').$$

Proof of Claim 2.1. Apply Poisson summation and simply compute the Fourier transform — the point being that $\mathrm{supp}\,\varphi_T + \mathrm{supp}\,\varphi_T \subseteq \left[-\frac{N_{\mathrm{all}}(T)}{T}, \frac{N_{\mathrm{all}}(T)}{T}\right]$ and so all but the zero mode of the relevant convolution vanishes. $\square$

Claim 2.2. *For $0 \leq k, \ell < N_{\mathrm{all}}(T)$,*

$$W_{k,\ell} = \frac{T}{N_{\mathrm{all}}(T) \cdot \int_{\mathbb{R}} \varphi_T^2} \left(\begin{array}{c} \int_{\mathbb{R}} d\tau\, \widehat{\varphi_T}\left(\tau - T - \frac{T}{N_{\mathrm{all}}(T)} k\right) \cdot \widehat{\varphi_T}\left(\tau - T - \frac{T}{N_{\mathrm{all}}(T)} \ell\right) \cdot \nu_T(\tau) \\ - \sum_{\rho \in \mathrm{far}_{N_{T^\delta}([T,2T))}} \mathrm{ord}_\rho\,\zeta \cdot \widehat{\varphi_T}\left(\gamma_\rho - T - \frac{T}{N_{\mathrm{all}}(T)} k\right) \cdot \widehat{\varphi_T}\left(\gamma_\rho - T - \frac{T}{N_{\mathrm{all}}(T)} \ell\right) \end{array} \right).$$

Proof of Claim 2.2. The claim is equivalently a calculation of

$$\sum_\rho \mathrm{ord}_\rho\,\zeta \cdot \widehat{\varphi_T}\left(\gamma_\rho - T - \frac{T}{N_{\mathrm{all}}(T)} \cdot k\right) \cdot \widehat{\varphi_T}\left(\gamma_\rho - T - \frac{T}{N_{\mathrm{all}}(T)} \cdot \ell\right),$$

and per the remark after our statement of Guinand-Weil it suffices to check that the relevant convolution of two φ_T's has support in $\left[-\frac{N_{\mathrm{all}}(T)}{T}, \frac{N_{\mathrm{all}}(T)}{T}\right]$, which it does. $\square$

Claim 2.3. $\mathrm{tr}\, W_T^{\mathrm{on}} \leq \sum_{\rho \in \mathrm{on}_{N_{T^\delta}([T,2T))}} \mathrm{ord}_\rho\,\zeta$ *and* $n_+\left(W_T^{\mathrm{off}}\right) \leq \frac{1}{2} \cdot \#|\mathrm{off}_{N_{T^\delta}([T,2T))}|$.

Proof of Claim 2.3. For $\rho \in \mathrm{on}_{N_{T^\delta}([T,2T))}$ one has $\gamma_\rho \in \mathbb{R}$, so

$$\mathrm{tr}\left(v_\rho \cdot v_\rho^t\right) = ||v_\rho||_2^2 = \sum_{k=0}^{N_{\mathrm{all}}(T)-1} \widehat{\varphi_T}\left(\gamma_\rho - T - \frac{T}{N_{\mathrm{all}}(T)} \cdot k\right)^2 \leq \frac{N_{\mathrm{all}}(T)}{T} \cdot \int_{\mathbb{R}} \varphi_T^2$$

by Claim 2.1 at $z = z' = \gamma_\rho$. Summing gives the trace bound.

For the rest, pair off $\rho \in \mathrm{off}_{N_{T^\delta}([T,2T))}$ as $(\rho, 1-\overline{\rho})$ via the functional equation. Because $\gamma_{1-\overline{\rho}} = \overline{\gamma_\rho}$ it follows that $v_{1-\overline{\rho}} = \overline{v_\rho}$ (Schwarz reflection), whence the desired hyperbolic plane i.e. $v_\rho \cdot v_\rho^t + \overline{v_\rho \cdot v_\rho^t}$. It follows that the relevant quadratic form is a pullback from $(1,1)^{\oplus \#|\mathrm{pairs}|}$ and the bound follows from comparing signatures. $\square$

3 Lemmas.

Lemma 3.1. $|\operatorname{tr} E_{\text{far}}|, ||E_{\text{far}}||_{\text{HS}} \ll_\chi T^{\frac{1}{2}-2\delta}$.

Proof of Lemma 3.1. Run the same calculation as in Claim 2.3 and apply the Paley-Wiener integration by parts bound and sum (the square-root coming from $|\,\mathfrak{Im}\,\gamma_\rho| < \frac{1}{2}$ a priori). □

Lemma 3.2. $\operatorname{tr} W_T = (1 + o_{\chi;T\to\infty}(1)) \cdot N_{\text{all}}(T)$.

Proof of Lemma 3.2. Again apply Paley-Wiener to control the contribution of $k \notin [0, N_{\text{all}}(T) - 1)$ by $\ll_\chi \sqrt{T} \cdot \frac{N_{\text{all}}(T)}{T} \cdot \operatorname{dist}(\mathfrak{Im}\,\rho, \{T, 2T\})^{-3}$ (and similarly for the $\ll T^\delta$ many ρ with $\mathfrak{Im}\,\rho$ within T^δ of the interval endpoints). □

Lemma 3.3. $||W_T||_{\text{HS}}^2 = \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2}} + o_{\chi;T\to\infty}(1)\right) \cdot N_{\text{all}}(T)$.

Proof of Lemma 3.3. The Hilbert-Schmidt norm is, after completing the relevant k-sum and localizing the relevant τ and τ' integrals to $[T, 2T)$, an integral of a kernel formed from a convolution of φ_T with itself against a product of ν_T with itself. ν_T has three terms: π_T, corresponding to the pole of ζ, is irrelevant here. The diagonal terms of the $\lambda_T \cdot \lambda_T$ product will contribute to the main term, and for the off-diagonal terms we will use Montgomery-Vaughan-Hilbert. $\mu \cdot \mu$ also contributes, and this is where we see the shape of the variational problem that ψ solves. All other cross-terms cancel (e.g. because of the cosine oscillation in λ_T).

Let $K(\tau, \tau') := \sum_{0\le k<N_{\text{all}}(T)} \widehat{\varphi}_T\left(\tau - T - \frac{T}{N_{\text{all}}(T)} \cdot k\right) \cdot \widehat{\varphi}_T\left(\tau' - T - \frac{T}{N_{\text{all}}(T)} \cdot k\right)$ and $\alpha_k := T + \frac{T}{N_{\text{all}}(T)} \cdot k$.

By definition of E_{far}, $\left(\frac{N_{\text{all}}(T)}{T} \cdot \int_{\mathbb{R}} \varphi_T^2\right)^2 \cdot ||W_T + E_{\text{far}}||_{\text{HS}}^2 = \int_{\mathbb{R}} d\tau \int_{\mathbb{R}} d\tau'\, K(\tau, \tau')^2 \cdot \nu_T(\tau) \cdot \nu_T(\tau')$. We will calculate the right-hand side and then use Lemma 3.1 to remove the E_{far} from the left-hand side. (Note that we expect the main term to be of order $T \cdot (\log T)^5$.)

First let us execute the Poisson summation à la Claim 2.1 and also localize the integrals to $[T, 2T]$.

Throughout the error terms below we freely write $\log T$ for $\frac{N_{\text{all}}(T)}{T}$ (and hence $\log\log T$ for $\log \frac{N_{\text{all}}(T)}{T}$); only main terms need the exact ratio. Record once: for real τ, τ' one has $|K(\tau, \tau')| \le \frac{N_{\text{all}}(T)}{T} \cdot \int_{\mathbb{R}} \varphi_T^2 \ll \log^2 T$ (Cauchy-Schwarz + Claim 2.1 at $z = z' = \tau$ resp. τ'); $\left|\widehat{\varphi_T^2}(s)\right| \le \int_{\mathbb{R}} \varphi_T^2 \ll \log T$; and $\int_{|s|\ge D} ds\, |\widehat{\varphi}_T(s)| \ll_\chi \min\left(\log\log T, \frac{1}{D}\right)$ (integrate the Paley-Wiener min).

Claim.
$$\int_{\mathbb{R}} d\tau \int_{\mathbb{R}} d\tau'\, K(\tau, \tau')^2 \cdot \nu_T(\tau) \cdot \nu_T(\tau') = \left(\frac{N_{\text{all}}(T)}{T}\right)^2 \cdot \int_T^{2T} d\tau \int_T^{2T} d\tau'\, \widehat{\varphi_T^2}(\tau - \tau')^2 \cdot \nu_T(\tau) \cdot \nu_T(\tau') + O_\chi\left(T \cdot \log^4 T \cdot \log\log T\right).$$

Proof. On $[T, 2T]^2$*:* by Claim 2.1, $K(\tau, \tau') - \frac{N_{\text{all}}(T)}{T} \cdot \widehat{\varphi_T^2}(\tau - \tau') = -\sum_{k\notin[0,N_{\text{all}}(T))} \widehat{\varphi}_T(\tau - \alpha_k) \cdot \widehat{\varphi}_T(\tau' - \alpha_k)$. Hence

$$\left|K(\tau, \tau')^2 - \left(\frac{N_{\text{all}}(T)}{T} \cdot \widehat{\varphi_T^2}(\tau - \tau')\right)^2\right| \ll \log^2 T \cdot \sum_{k\notin[0,N_{\text{all}}(T))} \left|\widehat{\varphi}_T(\tau - \alpha_k) \cdot \widehat{\varphi}_T(\tau' - \alpha_k)\right|.$$

Integrate against $|\nu_T(\tau) \cdot \nu_T(\tau')| \le T$ (Chebyshev + Stirling: $|\mu| \ll \log T$, $|\lambda_T| \ll \sqrt{T}$, $|\pi_T| \ll T^{-\frac{1}{2}}$ on $[T, 2T]$) over $[T, 2T]^2$; the k-th summand factors as $\left(\int_T^{2T} d\tau\, |\widehat{\varphi}_T(\tau - \alpha_k)|\right)^2$. For $k \notin [0, N_{\text{all}}(T))$ one has $\alpha_k \notin [T, 2T)$, so $|\tau - \alpha_k| \ge \frac{T}{N_{\text{all}}(T)} \cdot \operatorname{dist}(k, [0, N_{\text{all}}(T)))$ for all $\tau \in [T, 2T]$; hence $\int_T^{2T} d\tau\, |\widehat{\varphi}_T(\tau - \alpha_k)| \ll_\chi \min\left(\log\log T, \frac{\log T}{\operatorname{dist}(k,[0,N_{\text{all}}(T)))}\right)$. Summing the squares:

$$\sum_{k\notin[0,N_{\text{all}}(T))} \min\left(\log\log T, \frac{\log T}{\operatorname{dist}(k, \cdots)}\right)^2 \ll \log T \cdot \log\log T.$$

So this piece is $\ll_\chi T \cdot \log^3 T \cdot \log\log T$.

Off $[T, 2T]^2$*:* by symmetry $\le 2\int_{\tau\in\mathbb{R}} \int_{\tau'\notin[T,2T]}$. Pull one $|K| \ll \log^2 T$ out: $K(\tau, \tau')^2 \ll \log^2 T \cdot \sum_{0\le k<N_{\text{all}}(T)} |\widehat{\varphi}_T(\tau - \alpha_k) \cdot \widehat{\varphi}_T(\tau' - \alpha_k)|$. Factor the double integral:

$$\ll \log^2 T \cdot \sum_{0\le k<N_{\text{all}}(T)} \left(\int_{\mathbb{R}} d\tau\, |\widehat{\varphi}_T(\tau - \alpha_k)| \cdot |\nu_T|\right) \cdot \left(\int_{\mathbb{R}\setminus[T,2T]} d\tau'\, |\widehat{\varphi}_T(\tau' - \alpha_k)| \cdot |\nu_T|\right).$$

The first factor: $|\nu_T| \ll \sqrt{T}$ on $|\tau| \le T^2$, so $\le \sqrt{T} \cdot \int_{\mathbb{R}} |\widehat{\varphi}_T| \ll \sqrt{T} \cdot \log\log T$ (the $|\tau| > T^2$ tail is negligible via the $|\tau|^{-2}$ decay). The second factor: $\alpha_k \in [T, 2T]$, so for $\tau' \notin [T, 2T]$ one has $|\tau' - \alpha_k| \ge \frac{T}{N_{\text{all}}(T)} \cdot \min(k, N_{\text{all}}(T) - k)$; hence $\ll \sqrt{T} \cdot \min\left(\log\log T, \frac{\log T}{\min(k, N_{\text{all}}(T)-k)}\right)$. Summing:

$$\sum_{0 \le k < N_{\text{all}}(T)} \min\left(\log\log T, \frac{\log T}{\min(k, N_{\text{all}}(T) - k)}\right) \ll \log T \cdot \log N_{\text{all}}(T) \ll \log^2 T.$$

So this piece is $\ll_\chi \log^2 T \cdot \sqrt{T} \cdot \log\log T \cdot \sqrt{T} \cdot \log^2 T = T \cdot \log^4 T \cdot \log\log T$. □

Now let us split $\nu_T \cdot \nu_T = (\mu + \lambda_T + \pi_T)^2$. In each piece the error is at least one $\log T$ below the main term $\asymp T \cdot \log^3 T$.

Claim. $\int_T^{2T} d\tau \int_T^{2T} d\tau' \, \widehat{\varphi_T^2}(\tau - \tau')^2 \cdot \mu(\tau) \cdot \mu(\tau') = T \cdot \left(\frac{N_{\text{all}}(T)}{T}\right)^2 \cdot \int_{\mathbb{R}} \varphi_T^4 + O_\chi\left(T \cdot \log^2 T\right)$.

Proof. Write $\mu(\tau) = \frac{N_{\text{all}}(T)}{T} + O(1)$ on $[T, 2T]$ (Stirling: $\mu(\tau) = \frac{1}{2\pi} \log \frac{\tau}{2\pi} + O(\tau^{-1})$; and $\frac{N_{\text{all}}(T)}{T}$ is the average of μ over $[T, 2T]$ up to $O(\frac{\log T}{T})$ by Riemann-von Mangoldt) and $\mu(\tau) - \mu(\tau') \ll \frac{|\tau - \tau'|}{T}$ (since $\mu' \ll |\tau|^{-1}$). Freeze $\mu(\tau') \to \mu(\tau)$: cost $\ll \int_T^{2T} d\tau \, |\mu(\tau)| \cdot \frac{1}{T} \cdot \int_{\mathbb{R}} ds \, \widehat{\varphi_T^2}(s)^2 \cdot |s| \ll \log T \cdot \log\log T$ (via $\int_{\mathbb{R}} ds \, \widehat{\varphi_T^2}(s)^2 \cdot |s| \ll_\chi \log\log T$). Extend $\tau' \in [T, 2T] \to \tau' \in \mathbb{R}$: cost $\ll \int_T^{2T} d\tau \, \mu(\tau)^2 \cdot \int_{|s| \ge \text{dist}(\tau, \{T, 2T\})} ds \, \widehat{\varphi_T^2}(s)^2 \ll_\chi \log^2 T \cdot (\log T)^{2/3}$. Then $\int_{\mathbb{R}} ds \, \widehat{\varphi_T^2}(s)^2 = \int_{\mathbb{R}} \varphi_T^4$ (Plancherel), and $\int_T^{2T} \mu^2 = T \cdot \left(\frac{N_{\text{all}}(T)}{T}\right)^2 + O(N_{\text{all}}(T))$ (from $\mu = \frac{N_{\text{all}}(T)}{T} + O(1)$ squared). □

Claim. $\int_T^{2T} d\tau \int_T^{2T} d\tau' \, \widehat{\varphi_T^2}(\tau - \tau')^2 \cdot \lambda_T(\tau) \cdot \lambda_T(\tau') = 2T \cdot \int_0^{N_{\text{all}}(T)/T} dy \, y \cdot \left(\varphi_T^2 * \varphi_T^2\right)(y) + O_\chi\left(T \cdot \log^2 T\right)$.

Proof. Extend $\tau' \to \mathbb{R}$: cost $\ll |\lambda_T|_\infty^2 \cdot \int_{\tau \in [T, 2T], \tau' \notin [T, 2T]} \widehat{\varphi_T^2}(\tau - \tau')^2 \ll_\chi T \cdot (\log T)^{2/3}$ (via $|\lambda_T| \ll \sqrt{T}$ and $\int_{\tau \in [T, 2T], \tau' \notin [T, 2T]} \widehat{\varphi_T^2}(\tau - \tau')^2 = \int_0^T dD \int_{|s| \ge D} ds \, \widehat{\varphi_T^2}(s)^2 \ll_\chi (\log T)^{2/3}$).

Inner integral. Since $\int_{\mathbb{R}} ds \, \widehat{\varphi_T^2}(s)^2 \cdot e(sy) = \left(\varphi_T^2 * \varphi_T^2\right)(y)$ and $\widehat{\varphi_T^2}$ is even,

$$\int_{\mathbb{R}} d\tau' \, \widehat{\varphi_T^2}(\tau - \tau')^2 \cdot \lambda_T(\tau') = -\frac{1}{\pi} \sum_m \frac{\Lambda(m)}{\sqrt{m}} \cdot \left(\varphi_T^2 * \varphi_T^2\right)\left(\frac{\log m}{2\pi}\right) \cdot \cos(\tau \log m).$$

Multiply by $\lambda_T(\tau)$ and integrate over $\tau \in [T, 2T]$; write $\cos A \cdot \cos B = \frac{1}{2}(\cos(A + B) + \cos(A - B))$ with $A := \tau \log n$, $B := \tau \log m$.

Diagonal $n = m$. $\int_T^{2T} d\tau \, \cos^2(\tau \log n) = \frac{T}{2} + O\left(\frac{1}{\log n}\right)$, so this piece is $\frac{T}{2\pi^2} \sum_n \frac{\Lambda(n)^2}{n} \cdot \left(\varphi_T^2 * \varphi_T^2\right)\left(\frac{\log n}{2\pi}\right) + O_\chi(\log^2 T)$. Partial summation: with $A(t) := \sum_{n \le t} \frac{\Lambda(n)^2}{n} = \frac{1}{2} \log^2 t + O(\log t)$ and $g(y) := \left(\varphi_T^2 * \varphi_T^2\right)(y)$ (so $\operatorname{supp} g \subseteq \left[-\frac{N_{\text{all}}(T)}{T}, \frac{N_{\text{all}}(T)}{T}\right]$ and $||g'||_\infty \le ||(\varphi_T^2)'||_1 \ll_\chi 1$),

$$\sum_n \frac{\Lambda(n)^2}{n} \cdot g\left(\frac{\log n}{2\pi}\right) = -\int_0^{N_{\text{all}}(T)/T} dy \left(\frac{1}{2}(2\pi y)^2 + O(2\pi y)\right) \cdot g'(y) = 4\pi^2 \int_0^{N_{\text{all}}(T)/T} dy \, y \cdot g(y) + O_\chi(\log^2 T)$$

(the boundary term $A(T) \cdot g\left(\frac{\log T}{2\pi}\right) = 0$ since $\frac{\log T}{2\pi} > \frac{N_{\text{all}}(T)}{T}$; then integrate by parts once more). Times $\frac{T}{2\pi^2}$: $2T \int_0^{N_{\text{all}}(T)/T} dy \, y \cdot g(y) + O_\chi(T \cdot \log^2 T)$.

Off-diagonal $n \ne m$. $\int_T^{2T} d\tau \, \cos\left(\tau \log \frac{n}{m}\right) = \mathfrak{Im}\left(\frac{(n/m)^{2iT} - (n/m)^{iT}}{\log(n/m)}\right)$. Set $x_n := \frac{\Lambda(n)}{\sqrt{n}} \cdot n^{iT}$ and $x_n' := x_n \cdot g\left(\frac{\log n}{2\pi}\right)$ (respectively n^{2iT}): the sum is $\mathfrak{Im} \sum_{n \ne m} \frac{x_n \cdot \overline{x_m'}}{\log n - \log m}$, and Montgomery-Vaughan with $y_n := \log n$, $\sigma_n > \frac{1}{n+1}$ gives (in bilinear form: the theorem reads $-CD \le iH \le CD$, so $||D^{-\frac{1}{2}} \cdot iH \cdot D^{-\frac{1}{2}}||_{\text{op}} \le C$) $|\cdots| \ll \left(\sum_{n \le T} \Lambda(n)^2\right)^{\frac{1}{2}} \cdot ||g||_\infty \cdot \left(\sum_{m \le T} \Lambda(m)^2\right)^{\frac{1}{2}} \ll T \cdot \log^2 T$.

The $\cos(A + B)$*-half.* $\left|\int_T^{2T} d\tau \, \cos(\tau \log nm)\right| \le \frac{2}{\log 4}$; so this is $\ll \log T \cdot \left(\sum_n \frac{\Lambda(n)}{\sqrt{n}}\right) \cdot \left(\sum_m \frac{\Lambda(m)}{\sqrt{m}}\right) \ll T \cdot \log T$. □

Claim. $\int_T^{2T} d\tau \int_T^{2T} d\tau' \, \widehat{\varphi_T^2}(\tau - \tau')^2 \cdot \left(2\mu\lambda_T + 2\mu\pi_T + 2\lambda_T\pi_T + \pi_T^2\right) \ll_\chi T \cdot \log T$.

Proof. The three terms with π_T: bound $|\pi_T| \ll T^{-\frac{1}{2}}$ on $[T, 2T]$ and $\int_T^{2T} \int_T^{2T} \widehat{\varphi_T^2}(\tau - \tau')^2 \le T \cdot \int_{\mathbb{R}} \varphi_T^4 \ll T \cdot \log T$; then $|\mu\pi_T| \ll T^{-\frac{1}{2}} \log T$, $|\lambda_T\pi_T| \ll 1$, $|\pi_T^2| \ll T^{-1}$ give respectively $\ll \sqrt{T} \log^2 T$, $\ll T \log T$, $\ll \log T$.

The $2\mu\lambda_T$ term: extend $\tau' \to \mathbb{R}$ (cost $\ll \log T \cdot \sqrt{T} \cdot (\log T)^{2/3}$), then use the inner-integral formula from the previous Claim:

$$2\int_T^{2T} d\tau\, \mu(\tau) \cdot \left(-\frac{1}{\pi} \sum_m \frac{\Lambda(m)}{\sqrt{m}} \cdot g\left(\frac{\log m}{2\pi}\right) \cdot \cos\left(\tau \log m\right) \right).$$

Integrate by parts once in τ: $\int_T^{2T} d\tau\, \mu(\tau) \cdot \cos(\tau \log m) = \left[\frac{\mu(\tau)\sin(\tau\log m)}{\log m}\right]_T^{2T} - \int_T^{2T} d\tau\, \frac{\mu'(\tau)\sin(\cdots)}{\log m} \ll \frac{\log T}{\log m}$ (boundary $\ll \frac{\log T}{\log m}$; integral $\ll \frac{1}{\log m}$ via $\mu' \ll T^{-1}$). Then $\sum_m \frac{\Lambda(m)}{\sqrt{m}\cdot\log m} = \sum_{p\le T} \frac{1}{\sqrt{p}} + O(1) \ll \frac{\sqrt{T}}{\log T}$; times $||g||_\infty \ll \log T$ and the $\frac{\log T}{\log m}$: total $\ll \log^2 T \cdot \frac{\sqrt{T}}{\log T} \ll \sqrt{T} \cdot \log T$. ☐

Claim. $\left(\frac{N_{\text{all}}(T)}{T}\right)^2 \cdot \int_{\mathbb{R}} \varphi_T^4 + 2\int_0^{N_{\text{all}}(T)/T} dy\, y \cdot \left(\varphi_T^2 * \varphi_T^2\right)(y) = \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2}}\right) \cdot \frac{N_{\text{all}}(T)}{T} \cdot \left(\int_{\mathbb{R}} \varphi_T^2\right)^2 + O_\chi\left(\log^2 T\right)$.

Proof. Since $\psi'' + 2\psi = 0$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$, the map $u \mapsto \psi(u) + \int_{-\frac{1}{2}}^{\frac{1}{2}} dv\, |u-v| \cdot \psi(v)$ has vanishing second derivative there (differentiate under the integral: $\frac{d^2}{du^2}\int dv\, |u-v| \cdot \psi(v) = 2\psi(u)$), hence is affine, hence (being even) constant on $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Its value at $u = 0$ is $\psi(0) + 2\int_0^{\frac{1}{2}} dv\, v \cdot \psi(v) = \cos\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sin\frac{1}{\sqrt{2}}$; and $\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi = \sqrt{2}\sin\frac{1}{\sqrt{2}}$; the ratio is $\frac{1}{2} + \frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2}}$. Integrate the constant against ψ over $\left[-\frac{1}{2}, \frac{1}{2}\right]$:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi^2 + \int_{-\frac{1}{2}}^{\frac{1}{2}} du \int_{-\frac{1}{2}}^{\frac{1}{2}} dv\, |u-v| \cdot \psi(u) \cdot \psi(v) = \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2}}\right) \cdot \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi\right)^2;$$

the double integral equals $2\int_0^1 dw\, w \cdot (\psi * \psi)(w)$ (substitute $w = u - v$, use ψ even).

Rescale: $\varphi_T^2(x)$ and $\psi\left(\frac{xT}{N_{\text{all}}(T)}\right) \cdot \mathbf{1}_{|x| \le \frac{N_{\text{all}}(T)}{2T}}$ differ only on the two transition intervals of total length $O_\chi(1)$, on which both are ≤ 1. Hence $\int_{\mathbb{R}} \varphi_T^2 = \frac{N_{\text{all}}(T)}{T} \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi + O_\chi(1)$ and $\int_{\mathbb{R}} \varphi_T^4 = \frac{N_{\text{all}}(T)}{T} \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi^2 + O_\chi(1)$; and $\left(\varphi_T^2 * \varphi_T^2\right)(y) = \frac{N_{\text{all}}(T)}{T} \cdot (\psi * \psi)\left(\frac{yT}{N_{\text{all}}(T)}\right) + O_\chi(1)$ uniformly, so $\int_0^{N_{\text{all}}(T)/T} dy\, y \cdot (\varphi_T^2 * \varphi_T^2)(y) = \left(\frac{N_{\text{all}}(T)}{T}\right)^3 \cdot \int_0^1 dw\, w \cdot (\psi * \psi)(w) + O_\chi\left(\log^2 T\right)$. Substitute. ☐

Assemble: the $\mu\mu$ and $\lambda_T\lambda_T$ main terms sum to T times the left side of the last Claim, so

$$\int_T^{2T} d\tau \int_T^{2T} d\tau'\, \widehat{\varphi_T^2}(\tau - \tau')^2 \cdot \nu_T \cdot \nu_T = \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2}}\right) \cdot N_{\text{all}}(T) \cdot \left(\int_{\mathbb{R}} \varphi_T^2\right)^2 + O_\chi\left(T \cdot \log^2 T\right).$$

Multiply by $\left(\frac{N_{\text{all}}(T)}{T}\right)^2$, add the first Claim's error $O_\chi\left(T \cdot \log^4 T \cdot \log\log T\right)$, and divide through by $\left(\frac{N_{\text{all}}(T)}{T} \cdot \int_{\mathbb{R}} \varphi_T^2\right)^2 \asymp \log^4 T$: this gives $||W_T + E_{\text{far}}||_{\text{HS}}^2 = \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2}}\right) \cdot N_{\text{all}}(T) + O_\chi\left(T \cdot \log\log T\right)$. Finally $\left|\, ||W_T||_{\text{HS}}^2 - ||W_T + E_{\text{far}}||_{\text{HS}}^2 \right| \le 2 \cdot ||W_T + E_{\text{far}}||_{\text{HS}} \cdot ||E_{\text{far}}||_{\text{HS}} + ||E_{\text{far}}||_{\text{HS}}^2 \ll \sqrt{N_{\text{all}}(T)} \cdot T^{\frac{1}{2} - 2\delta}$ by Lemma 3.1.

☐

Lemma 3.4. *Let A and B be Hermitian forms on $\mathbb{C}^n$ with A positive semidefinite. Then:*

$$\operatorname{rank} A \ge 2 \cdot \operatorname{tr} A + 4 \cdot \operatorname{tr} B - 4n_+(B) - ||A + B||_{\text{HS}}^2.$$

Proof of Lemma 3.4. Write $r := \operatorname{rank} A$ and $F(A, B) := \operatorname{rank} A - 2 \cdot \operatorname{tr} A - 4 \cdot \operatorname{tr} B + 4n_+(B) + ||A + B||_{\text{HS}}^2$. The minima of $U \to \mathbb{R}$ via $U \mapsto F(A, U \cdot B \cdot U^\dagger) =: F(A, B_U)$ satisfy $\operatorname{tr}\left(A \cdot [B_U, H]\right) = 0 = \operatorname{tr}\left([A, B_U] \cdot H\right)$ for all Hermitian H, whence at a minimum $[A, B_U] = 0$. But then for the purpose of the inequality we may assume without loss of generality that A and B are diagonal, with eigenvalues $a_1 \ge \cdots \ge a_n \ge 0$ (so $a_i = 0$ for $i > r$) and $b_1, \ldots, b_n$. Then with $x_i := a_i + b_i$,

$$||A + B||_{\text{HS}}^2 = \sum_{i=1}^n x_i^2 \ge \sum_{i=1}^n 2a_i + 4b_i - c_i = 2\operatorname{tr} A + 4\operatorname{tr} B - \sum_i c_i,$$

where $c_i = 1, 0, 4$ according as $i \le r, b_i \le 0$; $i > r, b_i \le 0$; $b_i > 0$ (via $(x_i - 1)^2, x_i^2, (x_i - 2)^2 \ge 0$ and $-2b_i, -4b_i, 2a_i \ge 0$ respectively); and $\sum_{i=1}^n c_i \le r + 4 \cdot \#|\{i : b_i > 0\}| = r + 4\,n_+(B)$. Rearranging we conclude. ☐